Contents

Chapter 1

Functions

1.1 Review of Functions

1.1.1 A function is a rule which assigns each domain element to a unique range element. The independent variable is associated with the domain, while the dependent variable is associated with the range.

1.1.2 The independent variable belongs to the domain, while the dependent variable belongs to the range.

1.1.3 The vertical line test is used to determine whether a given graph represents a function. (Specifically, it tests whether the variable associated with the vertical axis is a function of the variable associated with the horizontal axis.) If every vertical line which intersects the graph does so in exactly one point, then the given graph represents a function. If any vertical line $x = a$ intersects the curve in more than one point, then there is more than one range value for the domain value $x = a$, so the given curve does not represent a function.

1.1.4 $f(2) = \frac{1}{2^3+1} = \frac{1}{9}$. $f(y^2) = \frac{1}{(y^2)^3+1} = \frac{1}{y^6+1}$.

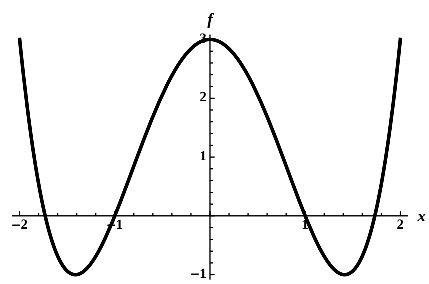
1.1.5 Item i. is true while item ii. isn't necessarily true. In the definition of function, item i. is stipulated. However, item ii. need not be true – for example, the function $f(x) = x^2$ has two different domain values associated with the one range value 4, because $f(2) = f(-2) = 4$.

1.1.6 $(f \circ g)(x) = f(g(x)) = f(x^3 - 2) = \sqrt{x^3 - 2}$
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = x^{3/2} - 2$.
$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$.
$(g \circ g)(x) = g(g(x)) = g(x^3 - 2) = (x^3 - 2)^3 - 2 = x^9 - 6x^6 + 12x^3 - 10$

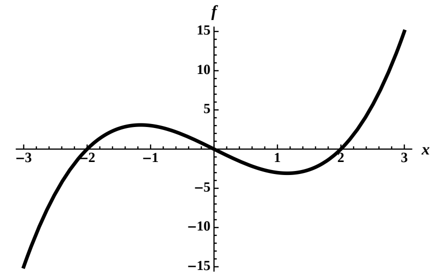
1.1.7 $f(g(2)) = f(-2) = f(2) = 2$. The fact that $f(-2) = f(2)$ follows from the fact that f is an even function.
$g(f(-2)) = g(f(2)) = g(2) = -2$.

1.1.8 The domain of $f \circ g$ is the subset of the domain of g whose range is in the domain of f. Thus, we need to look for elements x in the domain of g so that $g(x)$ is in the domain of f.

1.1.9 The defining property for an even function is that $f(-x) = f(x)$, which ensures that the graph of the function is symmetric about the y-axis.



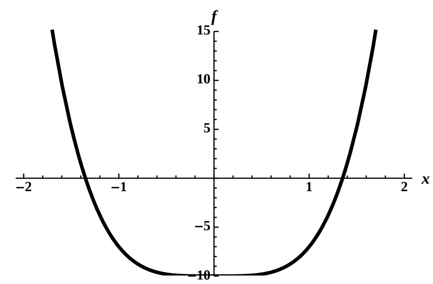
1.1.10 The defining property for an odd function is that $f(-x) = -f(x)$, which ensures that the graph of the function is symmetric about the origin.



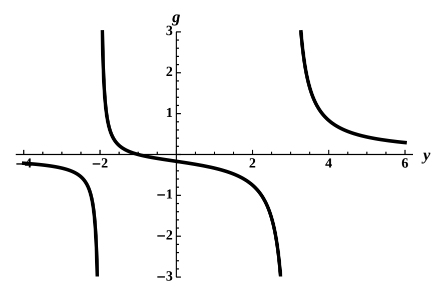
1.1.11 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

1.1.12 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

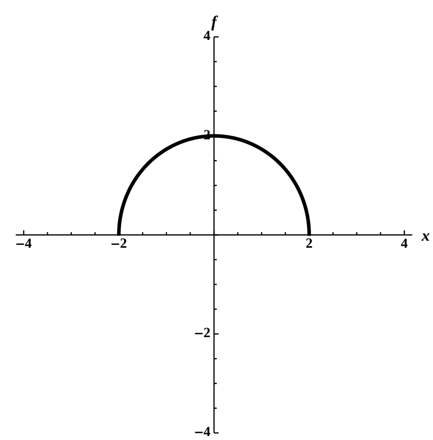
1.1.13 The natural domain of this function is the set of a real numbers. The range is $[-10, \infty)$.



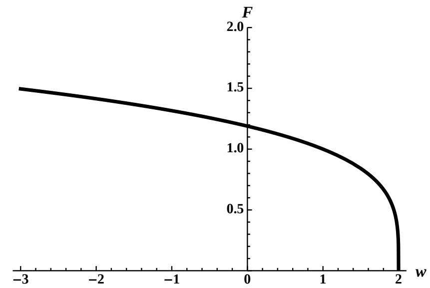
1.1.14 The natural domain of this function is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. The range is the set of all real numbers.



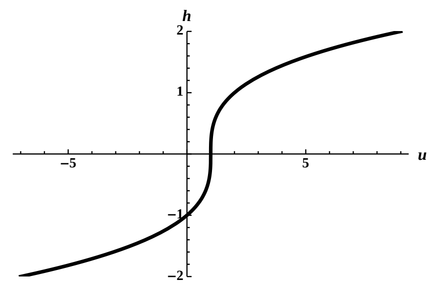
1.1.15 The natural domain of this function is $[-2, 2]$. The range is $[0, 2]$.



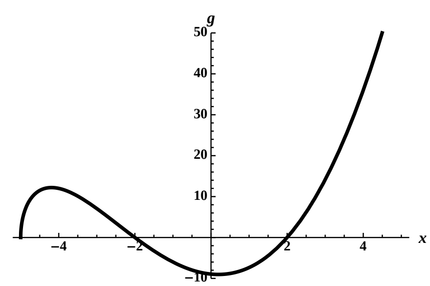
1.1.16 The natural domain of this function is $(-\infty, 2]$. The range is $[0, \infty)$.



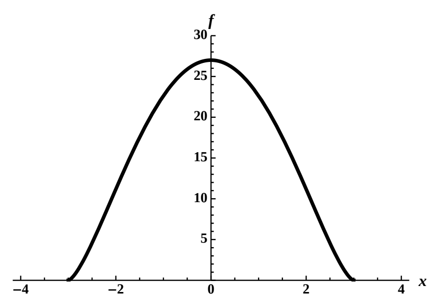
1.1.17 The natural domain and the range for this function are both the set of all real numbers.



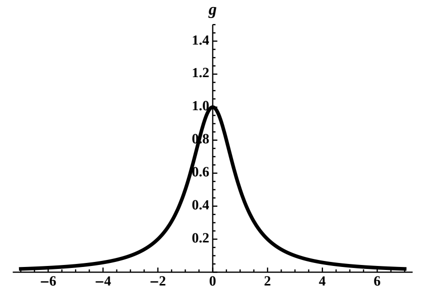
1.1.18 The natural domain of this function is $[-5, \infty)$. The range is approximately $[-9.03, \infty)$.



1.1.19 The natural domain of this function is $[-3, 3]$. The range is $[0, 27]$.



1.1.20 The natural domain of this function is $(-\infty, \infty)]$. The range is $(0, 1]$.



1.1.21 The independent variable t is elapsed time and the dependent variable d is distance above the ground. The domain in context is $[0, 8]$

1.1.22 The independent variable t is elapsed time and the dependent variable d is distance above the water. The domain in context is $[0, 2]$

1.1.23 The independent variable h is the height of the water in the tank and the dependent variable V is the volume of water in the tank. The domain in context is $[0, 50]$

1.1.24 The independent variable r is the radius of the balloon and the dependent variable V is the volume of the balloon. The domain in context is $[0, \sqrt[3]{3/(4\pi)}]$

1.1.25 $f(10) = 96$

1.1.26 $f(p^2) = (p^2)^2 - 4 = p^4 - 4$

1.1.27 $g(1/z) = (1/z)^3 = \frac{1}{z^3}$

1.1.28 $F(y^4) = \frac{1}{y^4-3}$

1.1.29 $F(g(y)) = F(y^3) = \frac{1}{y^3-3}$

1.1.30 $f(g(w)) = f(w^3) = (w^3)^2 - 4 = w^6 - 4$

1.1.31 $g(f(u)) = g(u^2 - 4) = (u^2 - 4)^3$

1.1.32 $\frac{f(2+h)-f(2)}{h} = \frac{(2+h)^2-4-0}{h} = \frac{4+4h+h^2-4}{h} = \frac{4h+h^2}{h} = 4 + h$

1.1.33 $F(F(x)) = F\left(\frac{1}{x-3}\right) = \frac{1}{\frac{1}{x-3}-3} = \frac{1}{\frac{1}{x-3}-\frac{3(x-3)}{x-3}} = \frac{1}{\frac{10-3x}{x-3}} = \frac{x-3}{10-3x}$

1.1.34 $g(F(f(x))) = g(F(x^2 - 4)) = g\left(\frac{1}{x^2-4-3}\right) = \left(\frac{1}{x^2-7}\right)^3$

1.1.35 $f(\sqrt{x+4}) = (\sqrt{x+4})^2 - 4 = x + 4 - 4 = x.$

1.1.36 $F((3x + 1)/x) = \frac{1}{\frac{3x+1}{x}-3} = \frac{1}{\frac{3x+1-3x}{x}} = \frac{x}{3x+1-3x} = x.$

1.1.37 $g(x) = x^3 - 5$ and $f(x) = x^{10}$. The domain of h is the set of all real numbers.

1.1.38 $g(x) = x^6 + x^2 + 1$ and $f(x) = \frac{2}{x^2}$. The domain of h is the set of all real numbers.

1.1.39 $g(x) = x^4 + 2$ and $f(x) = \sqrt{x}$. The domain of h is the set of all real numbers.

1.1.40 $g(x) = x^3 - 1$ and $f(x) = \frac{1}{\sqrt{x}}$. The domain of h is the set of all real numbers for which $x^3 - 1 > 0$, which corresponds to the set $(1, \infty)$.

1.1.41 $(f \circ g)(x) = f(g(x)) = f(x^2 - 4) = |x^2 - 4|$. The domain of this function is the set of all real numbers.

1.1.42 $(g \circ f)(x) = g(f(x)) = g(|x|) = |x|^2 - 4 = x^2 - 4$. The domain of this function is the set of all real numbers.

1.1.43 $(f \circ G)(x) = f(G(x)) = f\left(\frac{1}{x-2}\right) = \left|\frac{1}{x-2}\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.44 $(f \circ g \circ G)(x) = f(g(G(x))) = f\left(g\left(\frac{1}{x-2}\right)\right) = f\left(\left(\frac{1}{x-2}\right)^2 - 4\right) = \left|\left(\frac{1}{x-2}\right)^2 - 4\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.45 $(G \circ g \circ f)(x) = G(g(f(x))) = G(g(|x|)) = G(x^2 - 4) = \frac{1}{x^2-4-2} = \frac{1}{x^2-6}$. The domain of this function is the set of all real numbers except for the numbers $\pm\sqrt{6}$.

1.1.46 $(F \circ g \circ g)(x) = F(g(g(x))) = F(g(x^2 - 4)) = F((x^2 - 4)^2 - 4) = \sqrt{(x^2 - 4)^2 - 4} = \sqrt{x^4 - 8x^2 + 12}$. The domain of this function consists of the numbers x so that $x^4 - 8x^2 + 12 \geq 0$. Because $x^4 - 8x^2 + 12 = (x^2 - 6) \cdot (x^2 - 2)$, we see that this expression is zero for $x = \pm\sqrt{6}$ and $x = \pm\sqrt{2}$, By looking between these points, we see that the expression is greater than or equal to zero for the set $(-\infty, -\sqrt{6}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{2}, \infty)$.

1.1.47 $(g \circ g)(x) = g(g(x)) = g(x^2 - 4) = (x^2 - 4)^2 - 4 = x^4 - 8x^2 + 16 - 4 = x^4 - 8x^2 + 12$.

1.1.48 $(G \circ G)(x) = G(G(x)) = G(1/(x - 2)) = \frac{1}{\frac{1}{x-2} - 2} = \frac{1}{\frac{1-2(x-2)}{x-2}} = \frac{x-2}{1-2x+4} = \frac{x-2}{5-2x}$.

1.1.49 Because $(x^2 + 3) - 3 = x^2$, it must be the case that $f(x) = x - 3$.

1.1.50 Because the reciprocal of $x^2 + 3$ is $\frac{1}{x^2+3}$, it must be the case that $f(x) = \frac{1}{x}$.

1.1.51 Because $(x^2 + 3)^2 = x^4 + 6x^2 + 9$, it must be the case that $f(x) = x^2$.

1.1.52 Because $(x^2 + 3)^2 = x^4 + 6x^2 + 9$, and the given expression is 11 more than this, it must be the case that $f(x) = x^2 + 11$.

1.1.53 Because $(x^2)^2 + 3 = x^4 + 3$, this expression results from squaring x^2 and adding 3 to it. Thus we must have $f(x) = x^2$.

1.1.54 Because $x^{2/3} + 3 = (\sqrt[3]{x})^2 + 3$, we must have $f(x) = \sqrt[3]{x}$.

1.1.55

a. $f(g(2)) = f(2) = 4$

b. $g(f(2)) = g(4) = 1$

c. $f(g(4)) = f(1) = 3$

d. $g(f(5)) = g(6) = 3$

e. $f(g(7)) = f(4) = 7$

f. $f(f(8)) = f(8) = 8$

1.1.56

a. $h(g(0)) = h(0) = -1$

b. $g(f(4)) = g(-1) = -1$

c. $h(h(0)) = h(-1) = 0$

d. $g(h(f(4))) = g(h(-1)) = g(0) = 0$

e. $f(f(f(1))) = f(f(0)) = f(1) = 0$

f. $h(h(h(0))) = h(h(-1)) = h(0) = -1$

g. $f(h(g(2))) = f(h(3)) = f(0) = 1$

h. $g(f(h(4))) = g(f(4)) = g(-1) = -1$

i. $g(g(g(1))) = g(g(2)) = g(3) = 4$

j. $f(f(h(3))) = f(f(0)) = f(1) = 0$

1.1.57 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{(x^2+2hx+h^2)-x^2}{h} = \frac{h(2x+h)}{h} = 2x + h$.

$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = \frac{(x-a)(x+a)}{x-a} = x + a$.

1.1.58 $\frac{f(x+h)-f(x)}{h} = \frac{4(x+h)-3-(4x-3)}{h} = \frac{4x+4h-3-4x+3}{h} = \frac{4h}{h} = 4$.

$\frac{f(x)-f(a)}{x-a} = \frac{4x-3-(4a-3)}{x-a} = \frac{4x-4a}{x-a} = \frac{4(x-a)}{x-a} = 4$.

1.1.59 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{2}{x+h}-\frac{2}{x}}{h} = \frac{\frac{2x-2(x+h)}{x(x+h)}}{h} = \frac{2x-2x-2h}{(h)(x)(x+h)} = \frac{-2h}{(h)(x)(x+h)} = \frac{-2}{(x)(x+h)}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{2}{x}-\frac{2}{a}}{x-a} = \frac{\frac{2a-2x}{ax}}{x-a} = \frac{2(a-x)}{(x-a)(ax)} = \frac{-2(x-a)}{(x-a)(ax)} = \frac{-2}{ax}$.

1.1.60 $\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h} = \frac{2x^2+4xh+2h^2-3x-3h+1-2x^2+3x-1}{h} = \frac{4xh+2h^2-3h}{h} = \frac{h(4x+2h-3)}{h} = 4x+2h-3$.

$\frac{f(x)-f(a)}{x-a} = \frac{2x^2-3x+1-(2a^2-3a+1)}{x-a} = \frac{2(x^2-a^2)-3(x-a)}{x-a} = \frac{2(x-a)(x+a)-3(x-a)}{x-a} = \frac{(x-a)(2(x+a)-3)}{x-a} = 2(x+a)-3 = 2x+2a-3$.

1.1.61 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h} = \frac{\frac{(x+h)(x+1)-x(x+h+1)}{(x+1)(x+h+1)}}{h} = \frac{x^2+x+hx+h-x^2-xh-x}{(h)(x+1)(x+h+1)} = \frac{h}{(h)(x+1)(x+h+1)} = \frac{1}{(x+1)(x+h+1)}$

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{x}{x+1}-\frac{a}{a+1}}{x-a} = \frac{\frac{x(a+1)-a(x+1)}{(x+1)(a+1)}}{x-a} = \frac{xa+x-ax-a}{(x-a)(x+1)(a+1)} = \frac{x-a}{(x-a)(x+1)(a+1)} = \frac{1}{(x+1)(a+1)}$.

1.1.62 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^4-x^4}{h} = \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} = \frac{(h)(4x^3+6x^2h+4xh^2+h^3)}{h} = 4x^3+6x^2h+4xh^2+h^3$.

$\frac{f(x)-f(a)}{x-a} = \frac{x^4-a^4}{x-a} = \frac{(x^2-a^2)(x^2+a^2)}{x-a} = \frac{(x-a)(x+a)(x^2+a^2)}{x-a} = (x+a)(x^2+a^2)$.

1.1.63 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-2(x+h)-(x^3-2x)}{h} = \frac{x^3+3x^2h+3xh^2+h^3-2x-2h-x^3+2x}{h} = \frac{(h)(3x^2+3xh+h^2-2)}{h} = 3x^2+3xh+h^2-2$.

$\frac{f(x)-f(a)}{x-a} = \frac{x^3-2x-(a^3-2a)}{x-a} = \frac{(x^3-a^3)-2(x-a)}{x-a} = \frac{(x-a)(x^2+ax+a^2)-2(x-a)}{x-a} = \frac{(x-a)(x^2+ax+a^2-2)}{x-a} = x^2+ax+a^2-2$.

1.1.64 $\frac{f(x+h)-f(x)}{h} = \frac{4-4(x+h)-(x+h)^2-(4-4x-x^2)}{h} = \frac{4-4x-4h-x^2-2xh-h^2-4+4x+x^2}{h} = \frac{-4h-2xh-h^2}{h} = -4-2x-h$.

$\frac{f(x)-f(a)}{x-a} = \frac{4-4x-x^2-(4-4a-a^2)}{x-a} = \frac{-4(x-a)-(x^2-a^2)}{x-a} = \frac{-4(x-a)-(x-a)(x+a)}{x-a} = \frac{(x-a)(-4-(x+a))}{x-a} = -4-x-a$.

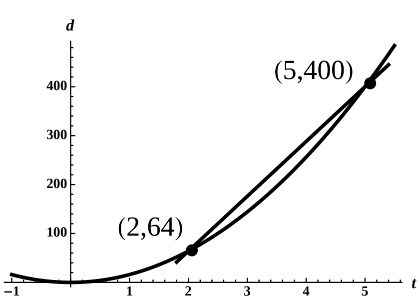
1.1.65 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{-4}{(x+h)^2}-\frac{-4}{x^2}}{h} = \frac{\frac{-4x^2+4(x+h)^2}{x^2(x+h)^2}}{h} = \frac{-4x^2+4x^2+8xh+4h^2}{x^2(x+h)^2(h)} = \frac{8x+4h}{x^2(x+h)^2} = \frac{4(2x+h)}{x^2(x+h)^2}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{-4}{x^2}-\frac{-4}{a^2}}{x-a} = \frac{\frac{-4a^2+4x^2}{a^2x^2}}{x-a} = \frac{4(x^2-a^2)}{(x-a)(a^2x^2)} = \frac{4(x-a)(x+a)}{(x-a)(a^2x^2)} = \frac{4(x+a)}{a^2x^2}$.

1.1.66 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-(x+h)^2-\left(\frac{1}{x}-x^2\right)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} - \frac{(x+h)^2-x^2}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} - \frac{x^2+2xh+h^2-x^2}{h} = \frac{-h}{(h)(x)(x+h)} - \frac{(h)(2x+h)}{h} = \frac{-1}{x(x+h)} - (2x+h)$.

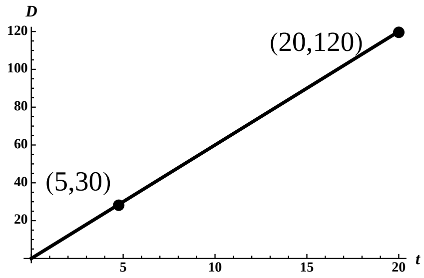
$\frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x}-x^2-(\frac{1}{a}-a^2)}{x-a} = \frac{\frac{1}{x}-\frac{1}{a}}{x-a} - \frac{x^2-a^2}{x-a} = \frac{\frac{a-x}{ax}}{x-a} - \frac{(x-a)(x+a)}{x-a} = \frac{-1}{ax} - (x+a)$.

1.1.67

a. 

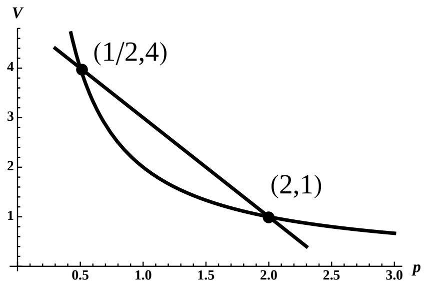
b. The slope of the secant line is given by $\frac{400-64}{5-2} = \frac{336}{3} = 112$ feet per second. The object falls at an average rate of 112 feet per second over the interval $2 \le t \le 5$.

1.1.68

a. 

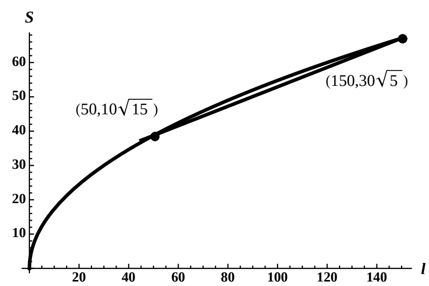
b. The slope of the secant line is given by $\frac{120-30}{20-5} = \frac{90}{15} = 6$ degrees per second. The second hand moves at an average rate of 6 degrees per second over the interval $5 \le t \le 20$.

1.1.69

a. 

b. The slope of the secant line is given by $\frac{1-4}{2-(1/2)} = \frac{-3}{3/2} = -2$ cubic cm per atmosphere. The volume decreases at an average rate of 2 cubic cm per atmosphere over the interval $0.5 \le p \le 2$.

1.1.70

a. 

b. The slope of the secant line is given by $\frac{30\sqrt{5}-10\sqrt{15}}{150-50} \approx .2835$ mph per foot. The speed of the car changes with an average rate of about .2835 mph per foot over the interval $50 \le l \le 150$.

1.1.71 This function is symmetric about the y-axis, because $f(-x) = (-x)^4 + 5(-x)^2 - 12 = x^4 + 5x^2 - 12 = f(x)$.

1.1.72 This function is symmetric about the origin, because $f(-x) = 3(-x)^5 + 2(-x)^3 - (-x) = -3x^5 - 2x^3 + x = -(3x^5 + 2x^3 - x) = f(x)$.

1.1.73 This function has none of the indicated symmetries. For example, note that $f(-2) = -26$, while $f(2) = 22$, so f is not symmetric about either the origin or about the y-axis, and is not symmetric about the x-axis because it is a function.

1.1.74 This function is symmetric about the y-axis. Note that $f(-x) = 2|-x| = 2|x| = f(x)$.

1.1.75 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $(-x)^{2/3} = ((-x)^2)^{1/3} = (x^2)^{1/3} = x^{2/3}$, and a similar fact holds for the term involving y.

1.1.76 This function is symmetric about the origin. Writing the function as $y = f(x) = x^{3/5}$, we see that $f(-x) = (-x)^{3/5} = -(x)^{3/5} = -f(x)$.

1.1.77 This function is symmetric about the origin. Note that $f(-x) = (-x)|(-x)| = -x|x| = -f(x)$.

1.1.78 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $|-x| = |x|$ and $|-y| = |y|$.

1.1.79 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the y-axis, so is even.

1.1.80 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the origin, so is odd.

1.1.81

a. True. A real number z corresponds to the domain element $z/2 + 19$, because $f(z/2 + 19) = 2(z/2 + 19) - 38 = z + 38 - 38 = z$.

b. False. The definition of function does not require that each range element comes from a unique domain element, rather that each domain element is paired with a unique range element.

c. True. $f(1/x) = \frac{1}{1/x} = x$, and $\frac{1}{f(x)} = \frac{1}{1/x} = x$.

d. False. For example, suppose that f is the straight line through the origin with slope 1, so that $f(x) = x$. Then $f(f(x)) = f(x) = x$, while $(f(x))^2 = x^2$.

e. False. For example, let $f(x) = x+2$ and $g(x) = 2x-1$. Then $f(g(x)) = f(2x-1) = 2x-1+2 = 2x+1$, while $g(f(x)) = g(x+2) = 2(x+2) - 1 = 2x + 3$.

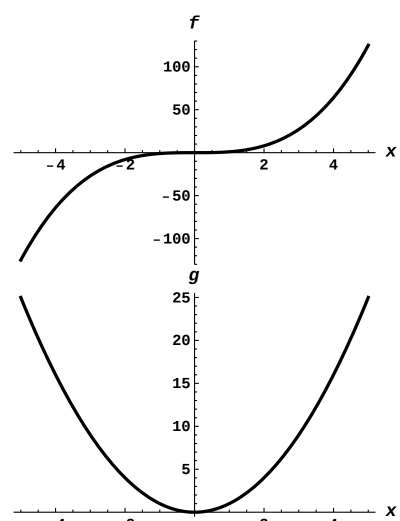
f. True. In fact, this is the definition of $f \circ g$.

g. True. If f is even, then $f(-z) = f(z)$ for all z, so this is true in particular for $z = ax$. So if $g(x) = cf(ax)$, then $g(-x) = cf(-ax) = cf(ax) = g(x)$, so g is even.

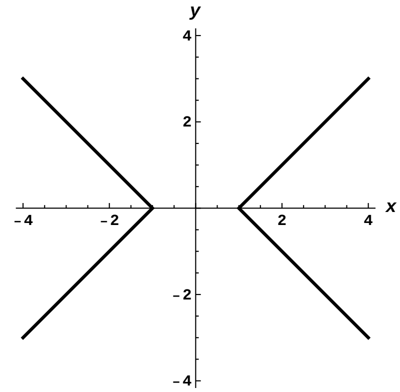
h. False. For example, $f(x) = x$ is an odd function, but $h(x) = x + 1$ isn't, because $h(2) = 3$, while $h(-2) = -1$ which isn't $-h(2)$.

i. True. If $f(-x) = -f(x) = f(x)$, then in particular $-f(x) = f(x)$, so $0 = 2f(x)$, so $f(x) = 0$ for all x.

1.1.82 If n is odd, then $n = 2k + 1$ for some integer k, and $(x)^n = (x)^{2k+1} = x(x)^{2k}$, which is less than 0 when $x < 0$ and greater than 0 when $x > 0$. For any number P (positive or negative) the number $\sqrt[n]{P}$ is a real number when n is odd, and $f(\sqrt[n]{P}) = P$. So the range of f in this case is the set of all real numbers.
If n is even, then $n = 2k$ for some integer k, and $x^n = (x^2)^k$. Thus $g(-x) = g(x) = (x^2)^k \geq 0$ for all x. Also, for any nonnegative number M, we have $g(\sqrt[n]{M}) = M$, so the range of g in this case is the set of all nonnegative numbers.



1.1.83 We will make heavy use of the fact that $|x|$ is x if $x > 0$, and is $-x$ if $x < 0$. In the first quadrant where x and y are both positive, this equation becomes $x - y = 1$ which is a straight line with slope 1 and y-intercept -1. In the second quadrant where x is negative and y is positive, this equation becomes $-x - y = 1$, which is a straight line with slope -1 and y-intercept -1. In the third quadrant where both x and y are negative, we obtain the equation $-x - (-y) = 1$, or $y = x + 1$, and in the fourth quadrant, we obtain $x + y = 1$. Graphing these lines and restricting them to the appropriate quadrants yields the following curve:



1.1.84

a. No. For example $f(x) = x^2 + 3$ is an even function, but $f(0)$ is not 0.

b. Yes. because $f(-x) = -f(x)$, and because $-0 = 0$, we must have $f(-0) = f(0) = -f(0)$, so $f(0) = -f(0)$, and the only number which is its own additive inverse is 0, so $f(0) = 0$.

1.1.85 Because the composition of f with itself has first degree, we can assume that f has first degree as well, so let $f(x) = ax+b$. Then $(f \circ f)(x) = f(ax+b) = a(ax+b)+b = a^2x+(ab+b)$. Equating coefficients, we see that $a^2 = 9$ and $ab + b = -8$. If $a = 3$, we get that $b = -2$, while if $a = -3$ we have $b = 4$. So two possible answers are $f(x) = 3x - 2$ and $f(x) = -3x + 4$.

1.1.86 Since the square of a linear function is a quadratic, we let $f(x) = ax+b$. Then $f(x)^2 = a^2x^2+2abx+b^2$. Equating coefficients yields that $a = \pm 3$ and $b = \pm 2$. However, a quick check shows that the middle term is correct only when one of these is positive and one is negative. So the two possible such functions f are $f(x) = 3x - 2$ and $f(x) = -3x + 2$.

1.1.87 Let $f(x) = ax^2 + bx + c$. Then $(f \circ f)(x) = f(ax^2 + bx + c) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$. Expanding this expression yields $a^3x^4 + 2a^2bx^3 + 2a^2cx^2 + ab^2x^2 + 2abcx + ac^2 + abx^2 + b^2x + bc + c$, which simplifies to $a^3x^4 + 2a^2bx^3 + (2a^2c + ab^2 + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c)$. Equating coefficients yields $a^3 = 1$, so $a = 1$. Then $2a^2b = 0$, so $b = 0$. It then follows that $c = -6$, so the original function was $f(x) = x^2 - 6$.

1.1.88 Because the square of a quadratic is a quartic, we let $f(x) = ax^2 + bx + c$. Then the square of f is $c^2 + 2bcx + b^2x^2 + 2acx^2 + 2abx^3 + a^2x^4$. By equating coefficients, we see that $a^2 = 1$ and so $a = \pm 1$. Because the coefficient on x^3 must be 0, we have that $b = 0$. And the constant term reveals that $c = \pm 6$. A quick check shows that the only possible solutions are thus $f(x) = x^2 - 6$ and $f(x) = -x^2 + 6$.

1.1.89 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}}.$

1.1.90 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} = \frac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2(x+h)}+\sqrt{1-2x}}{\sqrt{1-2(x+h)}+\sqrt{1-2x}} =$
$\frac{1-2(x+h)-(1-2x)}{(h)(\sqrt{1-2(x+h)}+\sqrt{1-2x})} = \frac{-2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}.$

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} \cdot \frac{\sqrt{1-2x}+\sqrt{1-2a}}{\sqrt{1-2x}+\sqrt{1-2a}} = \frac{(1-2x)-(1-2a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} =$
$\frac{(-2)(x-a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = \frac{-2}{(\sqrt{1-2x}+\sqrt{1-2a})}.$

1.1.91 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{-3}{\sqrt{x+h}}-\frac{-3}{\sqrt{x}}}{h} = \frac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} = \frac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \frac{-3(x-(x+h))}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{3}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{-3}{\sqrt{x}}-\frac{-3}{\sqrt{a}}}{x-a} = \frac{-3\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}\sqrt{x}}\right)}{x-a} = \frac{(-3)(\sqrt{a}-\sqrt{x})}{(x-a)\sqrt{a}\sqrt{x}} \cdot \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} = \frac{(3)(x-a)}{(x-a)(\sqrt{a}\sqrt{x})(\sqrt{a}+\sqrt{x})} = \frac{3}{\sqrt{ax}(\sqrt{a}+\sqrt{x})}$.

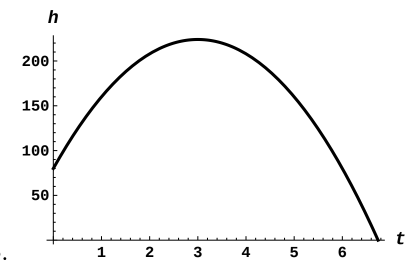
1.1.92 $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} = \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{(x+h)^2+1-(x^2+1)}{(h)(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \frac{x^2+2hx+h^2-x^2}{(h)(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$.

$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a} \cdot \frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}} = \frac{x^2+1-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1}}$.

1.1.93

a. The formula for the height of the rocket is valid from $t = 0$ until the rocket hits the ground, which is the positive solution to $-16t^2 + 96t + 80 = 0$, which the quadratic formula reveals is $t = 3 + \sqrt{14}$. Thus, the domain is $[0, 3 + \sqrt{14}]$.

b.



The maximum appears to occur at $t = 3$. The height at that time would be 224.

1.1.94

a. $d(0) = (10 - (2.2) \cdot 0)^2 = 100$.

b. The tank is first empty when $d(t) = 0$, which is when $10 - (2.2)t = 0$, or $t = 50/11$.

c. An appropriate domain would $[0, 50/11]$.

1.1.95 This would not necessarily have either kind of symmetry. For example, $f(x) = x^2$ is an even function and $g(x) = x^3$ is odd, but the sum of these two is neither even nor odd.

1.1.96 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $(f \cdot g)(-x) = f(-x)g(-x) = f(x) \cdot (-g(x)) = -(f \cdot g)(x)$.

1.1.97 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $\frac{f}{g}(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f}{g}(x)$.

1.1.98 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $f(g(-x)) = f(-g(x)) = f(g(x))$.

1.1.99 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is even. Then $f(g(-x)) = f(g(x))$, because $g(-x) = g(x)$.

1.1.100 This would be an odd function, so it would be symmetric about the origin. Suppose f is odd and g is odd. Then $f(g(-x)) = f(-g(x)) = -f(g(x))$.

1.1.101 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $g(f(-x)) = g(f(x))$, because $f(-x) = f(x)$.

1.1.102

a. $f(g(-1)) = f(-g(1)) = f(3) = 3$

b. $g(f(-4)) = g(f(4)) = g(-4) = -g(4) = 2$

c. $f(g(-3)) = f(-g(3)) = f(4) = -4$

d. $f(g(-2)) = f(-g(2)) = f(1) = 2$

e. $g(g(-1)) = g(-g(1)) = g(3) = -4$

f. $f(g(0) - 1) = f(-1) = f(1) = 2$

g. $f(g(g(-2))) = f(g(-g(2))) = f(g(1)) = f(-3) = 3$

h. $g(f(f(-4))) = g(f(-4)) = g(-4) = 2$

i. $g(g(g(-2))) = g(g(3)) = g(-4) = 2$

1.1.103

a. $f(g(-2) = f(-g(2)) = f(-2) = 4$

b. $g(f(-2)) = g(f(2)) = g(4) = 1$

c. $f(g(-4)) = f(-g(4)) = f(-1) = 3$

d. $g(f(5) - 8) = g(-2) = -g(2) = -2$

e. $g(g(-7)) = g(-g(7)) = g(-4) = -1$

f. $f(1 - f(8)) = f(-7) = 7$

1.2 Representing Functions

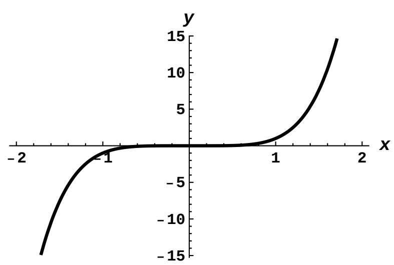
1.2.1 Functions can be defined and represented by a formula, through a graph, via a table, and by using words.

1.2.2 The domain of every polynomial is the set of all real numbers.

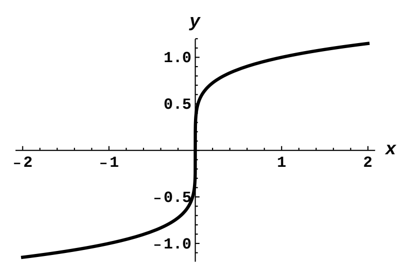
1.2.3 The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers for which $q(x) \neq 0$.

1.2.4 A piecewise linear function is one which is linear over intervals in the domain.

1.2.5



1.2.6



1.2.7 Compared to the graph of $f(x)$, the graph of $f(x+2)$ will be shifted 2 units to the left.

1.2.8 Compared to the graph of $f(x)$, the graph of $-3f(x)$ will be stretched vertically by a factor of 3 and flipped about the x axis.

1.2.9 Compared to the graph of $f(x)$, the graph of $f(3x)$ will be scaled horizontally by a factor of 3.

1.2.10 To produce the graph of $y = 4(x+3)^2 + 6$ from the graph of x^2, one must

1. shift the graph horizontally by 3 units to left

2. scale the graph vertically by a factor of 4

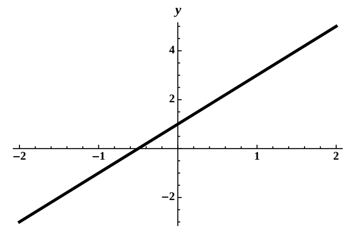
3. shift the graph vertically up 6 units.

1.2.11 The slope of the line shown is $m = \frac{-3-(-1)}{3-0} = -2/3$. The y-intercept is $b = -1$. Thus the function is given by $f(x) = (-2/3)x - 1$.

1.2.12 The slope of the line shown is $m = \frac{1-(5)}{5-0} = -4/5$. The y-intercept is $b = 5$. Thus the function is given by $f(x) = (-4/5)x + 5$.

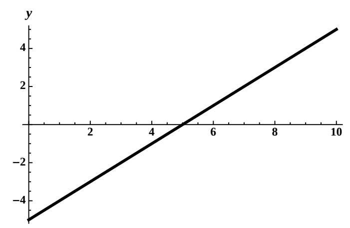
1.2.13

The slope is given by $\frac{5-3}{2-1} = 2$, so the equation of the line is $y - 3 = 2(x - 1)$, which can be written as $y = 2x - 2 + 3$, or $y = 2x + 1$.

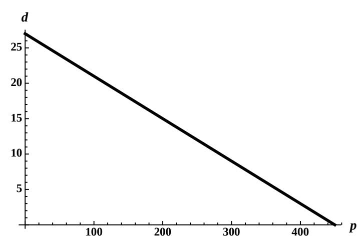


1.2.14

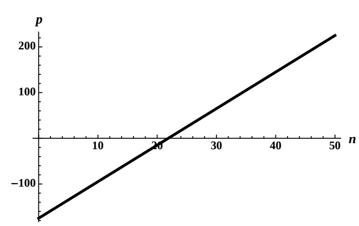
The slope is given by $\frac{0-(-3)}{5-2} = 1$, so the equation of the line is $y - 0 = 1(x - 5)$, or $y = x - 5$.



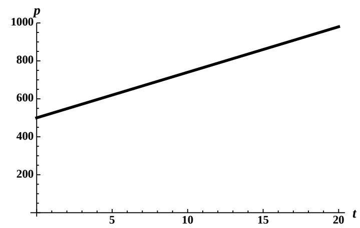
1.2.15 Using price as the independent variable p and the average number of units sold per day as the dependent variable d, we have the ordered pairs $(250, 12)$ and $(200, 15)$. The slope of the line determined by these points is $m = \frac{15-12}{200-250} = \frac{3}{-50}$. Thus the demand function has the form $d(p) = (-3/50)p + b$ for some constant b. Using the point $(200, 15)$, we find that $15 = (-3/50) \cdot 200 + b$, so $b = 27$. Thus the demand function is $d = (-3/50)p + 27$. While the natural domain of this linear function is the set of all real numbers, the formula is only likely to be valid for some subset of the interval $(0, 450)$, because outside of that interval either $p \leq 0$ or $d \leq 0$.



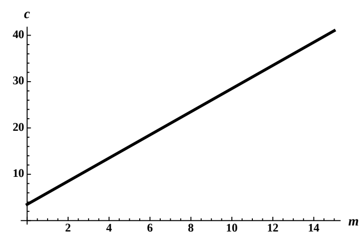
1.2.16 The profit is given by $p = f(n) = 8n - 175$. The break-even point is when $p = 0$, which occurs when $n = 175/8 = 21.875$, so they need to sell at least 22 tickets to not have a negative profit.



1.2.17 The slope is given by the rate of growth, which is 24. When $t = 0$ (years past 2010), the population is 500, so the point $(0, 500)$ satisfies our linear function. Thus the population is given by $p(t) = 24t + 500$. In 2025, we have $t = 15$, so the population will be approximately $p(15) = 360 + 500 = 860$.



1.2.18 The cost per mile is the slope of the desired line, and the intercept is the fixed cost of 3.5. Thus, the cost per mile is given by $c(m) = 2.5m + 3.5$. When $m = 9$, we have $c(9) = (2.5)(9) + 3.5 = 22.5 + 3.5 = 26$ dollars.



1.2.19 For $x < 0$, the graph is a line with slope 1 and y- intercept 3, while for $x > 0$, it is a line with slope $-1/2$ and y-intercept 3. Note that both of these lines contain the point $(0, 3)$. The function shown can thus be written

$$f(x) = \begin{cases} x + 3 & \text{if } x \leq 0; \\ (-1/2)x + 3 & \text{if } x > 0. \end{cases}$$

1.2.20 For $x < 3$, the graph is a line with slope 1 and y- intercept 1, while for $x > 3$, it is a line with slope $-1/3$. The portion to the right thus is represented by $y = (-1/3)x + b$, but because it contains the point $(6, 1)$, we must have $1 = (-1/3)(6) + b$ so $b = 3$. The function shown can thus be written

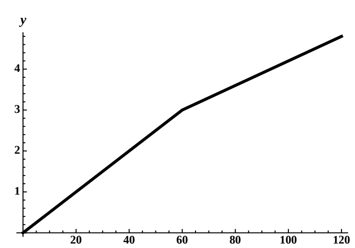
$$f(x) = \begin{cases} x + 1 & \text{if } x < 3; \\ (-1/3)x + 3 & \text{if } x \geq 3. \end{cases}$$

Note that at $x = 3$ the value of the function is 2, as indicated by our formula.

1.2.21

The cost is given by

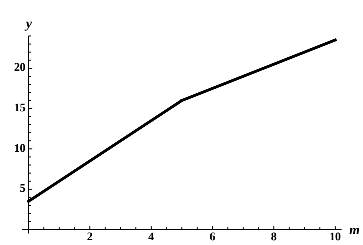
$$c(t) = \begin{cases} 0.05t & \text{for } 0 \leq t \leq 60 \\ 1.2 + 0.03t & \text{for } 60 < t \leq 120 \end{cases}.$$



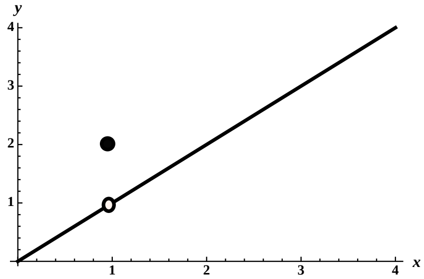
1.2.22

The cost is given by

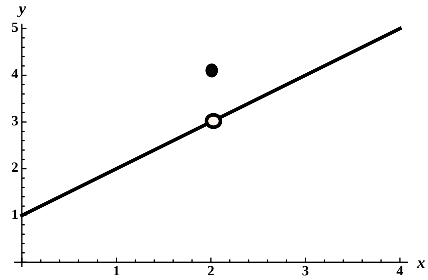
$$c(m) = \begin{cases} 3.5 + 2.5m & \text{for } 0 \leq m \leq 5 \\ 8.5 + 1.5m & \text{for } m > 5 \end{cases}.$$



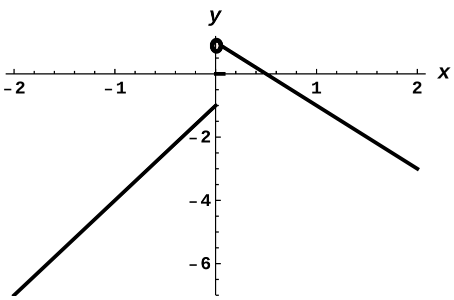
1.2.23



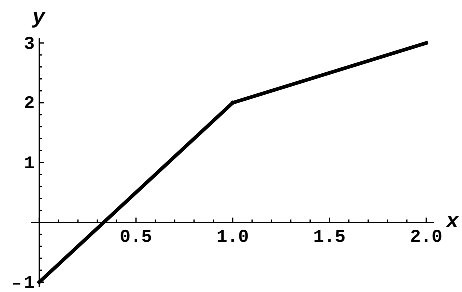
1.2.24



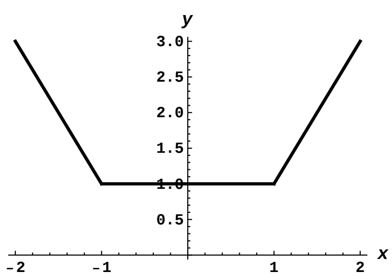
1.2.25



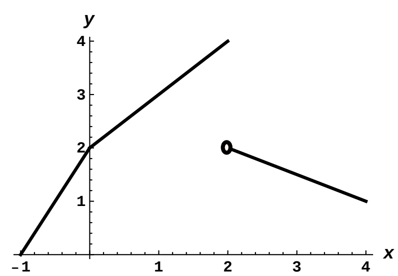
1.2.26



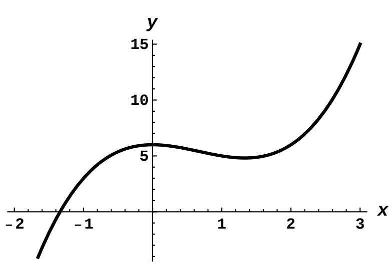
1.2.27



1.2.28



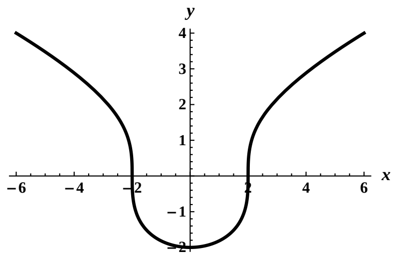
1.2.29

a. 

b. The function is a polynomial, so its domain is the set of all real numbers.

c. It has one peak near its y-intercept of $(0, 6)$ and one valley between $x = 1$ and $x = 2$. Its x-intercept is near $x = -4/3$.

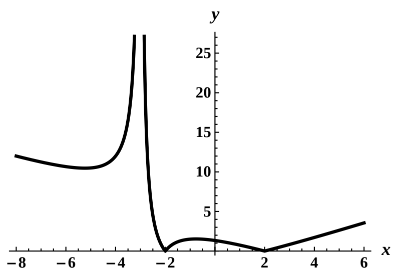
1.2.30

a. 

b. The function is an algebraic function. Its domain is the set of all real numbers.

c. It has a valley at the y-intercept of $(0, -2)$, and is very steep at $x = -2$ and $x = 2$ which are the x-intercepts. It is symmetric about the y-axis.

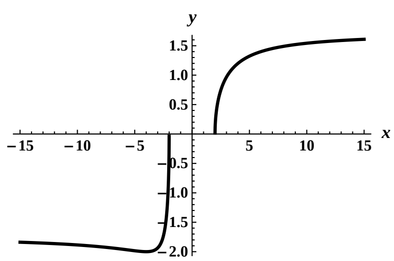
1.2.31

a. 

b. The domain of the function is the set of all real numbers except -3.

c. There is a valley near $x = -5.2$ and a peak near $x = -0.8$. The x-intercepts are at -2 and 2, where the curve does not appear to be smooth. There is a vertical asymptote at $x = -3$. The function is never below the x-axis. The y-intercept is $(0, 4/3)$.

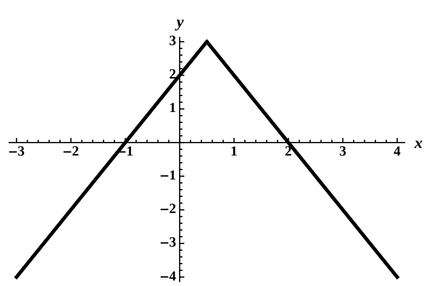
1.2.32

a. 

b. The domain of the function is $(-\infty, -2] \cup [2, \infty)$

c. x-intercepts are at -2 and 2. Because 0 isn't in the domain, there is no y-intercept. The function has a valley at $x = -4$.

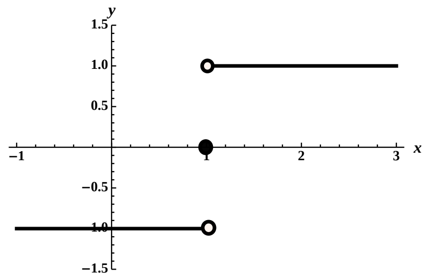
1.2.33

a. 

b. The domain of the function is $(-\infty, \infty)$

c. The function has a maximum of 3 at $x = 1/2$, and a y-intercept of 2.

1.2.34

a. 

b. The domain of the function is $(-\infty, \infty)$

c. The function contains a jump at $x = 1$. The maximum value of the function is 1 and the minimum value is -1.

1.2.35 The slope of this line is constantly 2, so the slope function is $s(x) = 2$.

1.2.36 The function can be written as $|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$.

The slope function is $s(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

1.2.37 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 0; \\ -1/2 & \text{if } x > 0. \end{cases}$

1.2.38 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 3; \\ -1/3 & \text{if } x > 3. \end{cases}$

1.2.39

a. Because the area under consideration is that of a rectangle with base 2 and height 6, $A(2) = 12$.

b. Because the area under consideration is that of a rectangle with base 6 and height 6, $A(6) = 36$.

c. Because the area under consideration is that of a rectangle with base x and height 6, $A(x) = 6x$.

1.2.40

a. Because the area under consideration is that of a triangle with base 2 and height 1, $A(2) = 1$.

b. Because the area under consideration is that of a triangle with base 6 and height 3, the $A(6) = 9$.

c. Because $A(x)$ represents the area of a triangle with base x and height $(1/2)x$, the formula for $A(x)$ is $\frac{1}{2} \cdot x \cdot \frac{x}{2} = \frac{x^2}{4}$.

1.2.41

a. Because the area under consideration is that of a trapezoid with base 2 and heights 8 and 4, we have $A(2) = 2 \cdot \frac{8+4}{2} = 12$.

b. Note that $A(3)$ represents the area of a trapezoid with base 3 and heights 8 and 2, so $A(3) = 3 \cdot \frac{8+2}{2} = 15$. So $A(6) = 15 + (A(6) - A(3))$, and $A(6) - A(3)$ represents the area of a triangle with base 3 and height 2. Thus $A(6) = 15 + 6 = 21$.

c. For x between 0 and 3, $A(x)$ represents the area of a trapezoid with base x, and heights 8 and $8-2x$. Thus the area is $x \cdot \frac{8+8-2x}{2} = 8x - x^2$. For $x > 3$, $A(x) = A(3) + A(x) - A(3) = 15 + 2(x-3) = 2x + 9$. Thus

$$A(x) = \begin{cases} 8x - x^2 & \text{if } 0 \leq x \leq 3; \\ 2x + 9 & \text{if } x > 3. \end{cases}$$

1.2.42

a. Because the area under consideration is that of trapezoid with base 2 and heights 3 and 1, we have $A(2) = 2 \cdot \frac{3+1}{2} = 4$.

b. Note that $A(6) = A(2) + A(6) - A(2)$, and that $A(6) - A(2)$ represents a trapezoid with base $6 - 2 = 4$ and heights 1 and 5. The area is thus $4 + \left(4 \cdot \frac{1+5}{2}\right) = 4 + 12 = 16$.

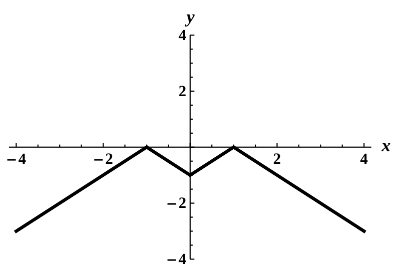
c. For x between 0 and 2, $A(x)$ represents the area of a trapezoid with base x, and heights 3 and $3 - x$. Thus the area is $x \cdot \frac{3+3-x}{2} = 3x - \frac{x^2}{2}$. For $x > 2$, $A(x) = A(2) + A(x) - A(2) = 4 + (A(x) - A(2))$. Note that $A(x) - A(2)$ represents the area of a trapezoid with base $x - 2$ and heights 1 and $x - 1$. Thus $A(x) = 4 + (x-2) \cdot \frac{1+x-1}{2} = 4 + (x-2)\left(\frac{x}{2}\right) = \frac{x^2}{2} - x + 4$. Thus

$$A(x) = \begin{cases} 3x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 2; \\ \frac{x^2}{2} - x + 4 & \text{if } x > 2. \end{cases}$$

1.2.43 $f(x) = |x-2| + 3$, because the graph of f is obtained from that of $|x|$ by shifting 2 units to the right and 3 units up.

$g(x) = -|x+2| - 1$, because the graph of g is obtained from the graph of $|x|$ by shifting 2 units to the left, then reflecting about the x-axis, and then shifting 1 unit down.

1.2.44

a.


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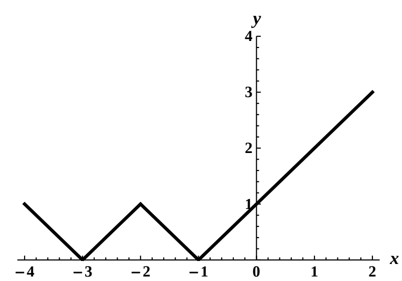
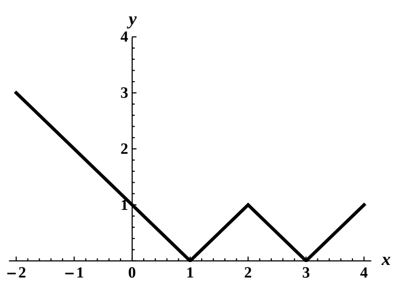
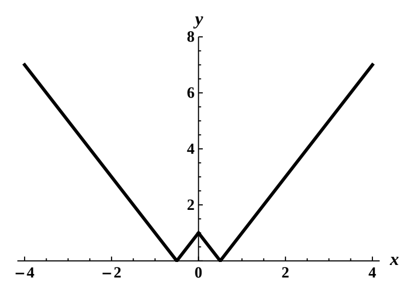
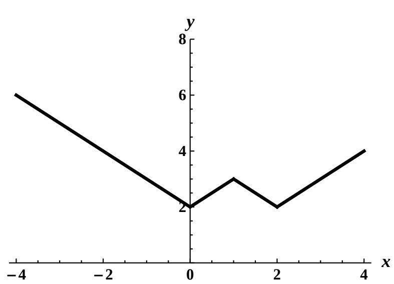
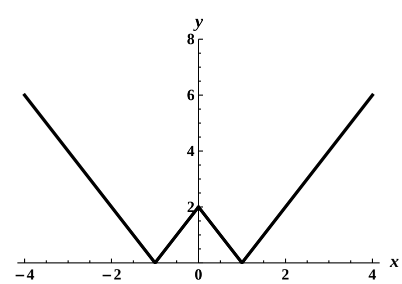


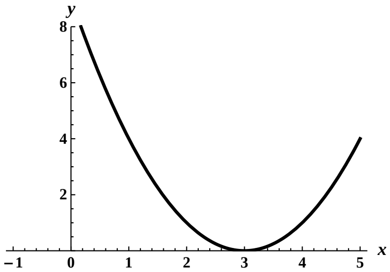
c.


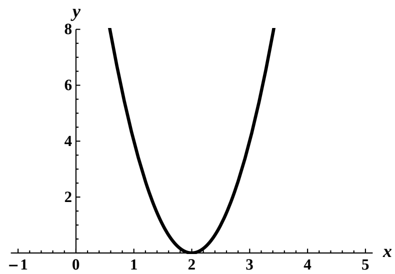
d.


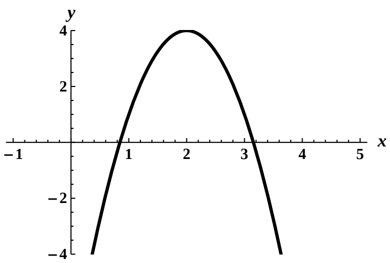
e.


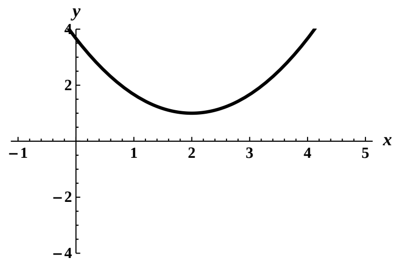
f.


1.2.45

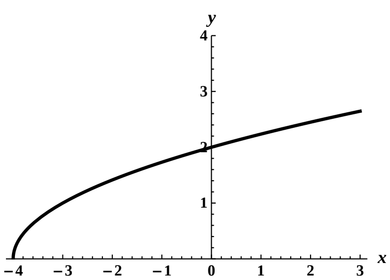
a. 

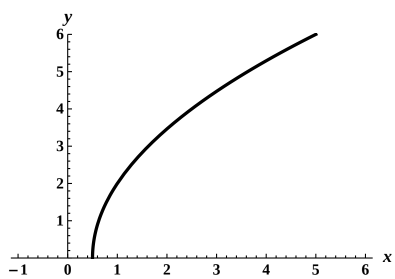
b. 

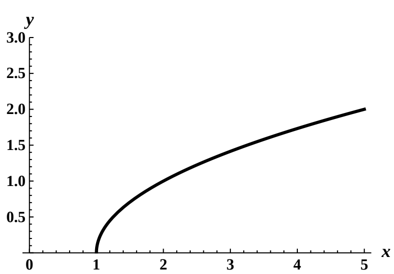
c. 

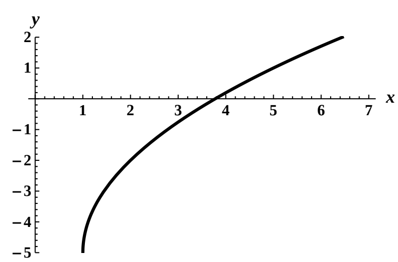
d. 

1.2.46

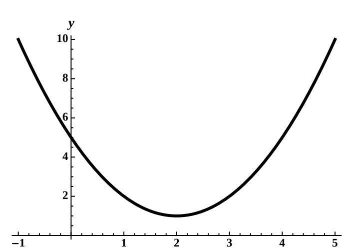
a. 

b. 

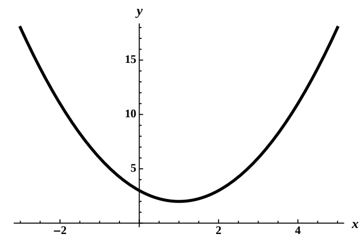
c. 

d. 

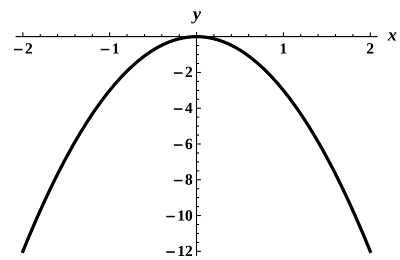
1.2.47 The graph is obtained by shifting the graph of x^2 two units to the right and one unit up.



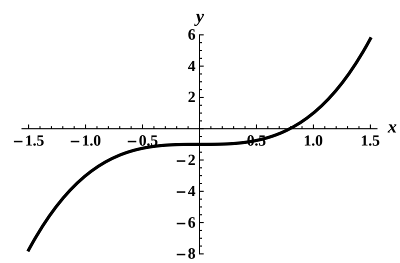
1.2.48 Write x^2-2x+3 as $(x^2-2x+1)+2=(x-1)^2+2$. The graph is obtained by shifting the graph of x^2 one unit to the right and two units up.



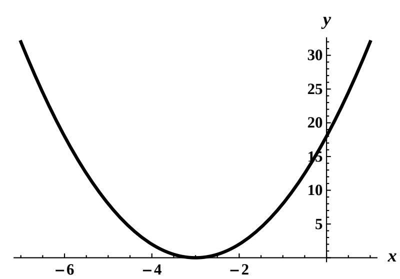
1.2.49 This function is $-3 \cdot f(x)$ where $f(x)=x^2$



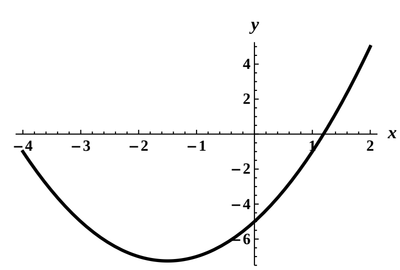
1.2.50 This function is $2 \cdot f(x)-1$ where $f(x)=x^3$



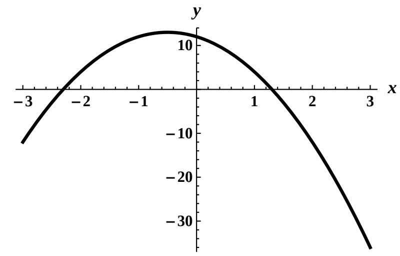
1.2.51 This function is $2 \cdot f(x+3)$ where $f(x)=x^2$



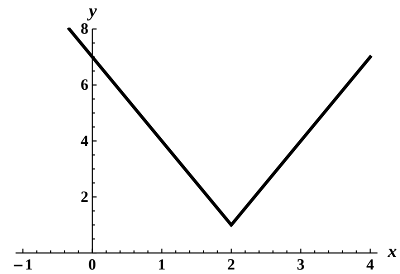
1.2.52 By completing the square, we have that $p(x) = (x^2+3x+(9/4))-(29/4)=(x+(3/2))^2-(29/4)$. So it is $f(x+(3/2))-(29/4)$ where $f(x)=x^2$.



1.2.53 By completing the square, we have that $h(x) = -4(x^2 + x - 3) = -4\left(x^2 + x + \frac{1}{4} - \frac{1}{4} - 3\right) = -4(x + (1/2))^2 + 13$. So it is $-4f(x + (1/2)) + 13$ where $f(x) = x^2$.



1.2.54 Because $|3x-6|+1 = 3|x-2|+1$, this is $3f(x-2)+1$ where $f(x) = |x|$.



1.2.55

a. True. A polynomial $p(x)$ can be written as the ratio of polynomials $\frac{p(x)}{1}$, so it is a rational function. However, a rational function like $\frac{1}{x}$ is not a polynomial.

b. False. For example, if $f(x) = 2x$, then $(f \circ f)(x) = f(f(x)) = f(2x) = 4x$ is linear, not quadratic.

c. True. In fact, if f is degree m and g is degree n, then the degree of the composition of f and g is $m \cdot n$, regardless of the order they are composed.

d. False. The graph would be shifted two units to the left.

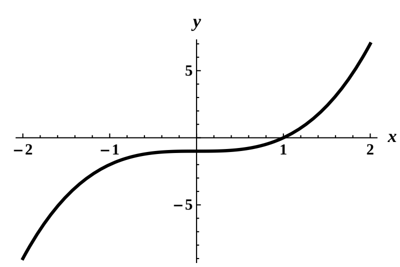
1.2.56 The points of intersection are found by solving $x^2 + 2 = x + 4$. This yields the quadratic equation $x^2 - x - 2 = 0$ or $(x - 2)(x + 1) = 0$. So the x-values of the points of intersection are 2 and -1. The actual points of intersection are $(2, 6)$ and $(-1, 3)$.

1.2.57 The points of intersection are found by solving $x^2 = -x^2 + 8x$. This yields the quadratic equation $2x^2 - 8x = 0$ or $(2x)(x - 4) = 0$. So the x-values of the points of intersection are 0 and 4. The actual points of intersection are $(0, 0)$ and $(4, 16)$.

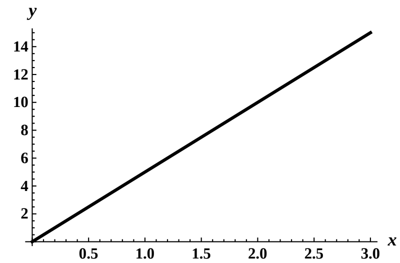
1.2.58 $y = x + 1$, because the y value is always 1 more than the x value.

1.2.59 $y = \sqrt{x} - 1$, because the y value is always 1 less than the square root of the x value.

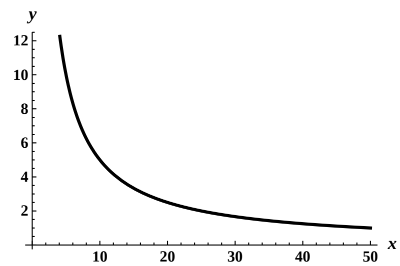
1.2.60 $y = x^3 - 1$. The domain is $(-\infty, \infty)$.



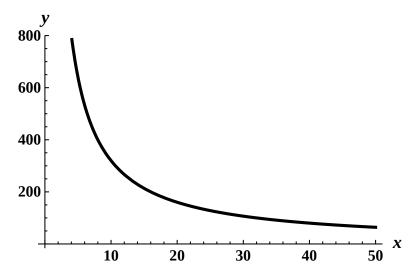
1.2.61 $y = 5x$. The natural domain for the situation is $[0, h]$ where h represents the maximum number of hours that you can run at that pace before keeling over.



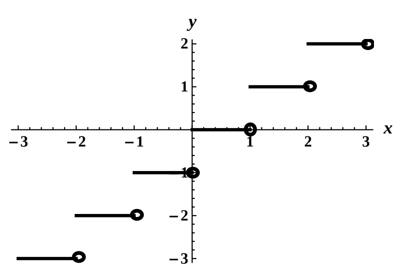
1.2.62 $y = \frac{50}{x}$. Theoretically the domain is $(0, \infty)$, but the world record for the "hour ride" is just short of 50 miles.



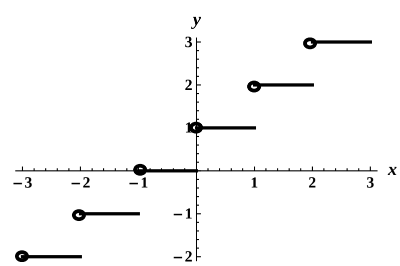
1.2.63 $y = \frac{3200}{x}$. Note that $\frac{x \text{ dollars per gallon}}{32 \text{ miles per gallon}} \cdot y \text{ miles}$ would represent the numbers of dollars, so this must be 100. So we have $\frac{xy}{32} = 100$, or $y = \frac{3200}{x}$. We certainly have $x > 0$, but unfortunately, there appears to be no no upper bound for x, so the domain is $(0, \infty)$.



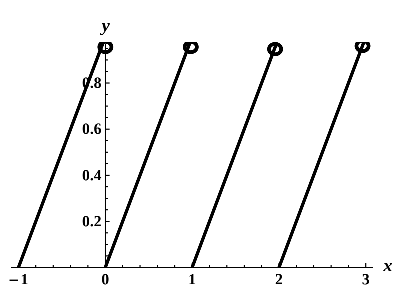
1.2.64



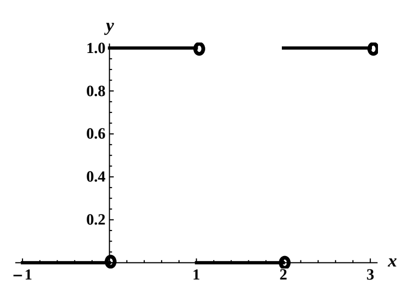
1.2.65



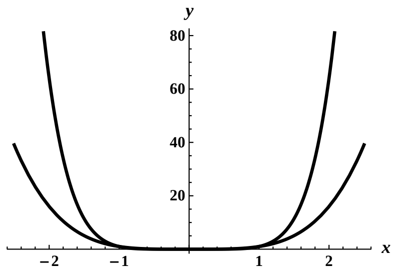
1.2.66



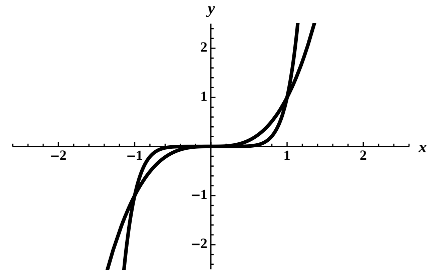
1.2.67



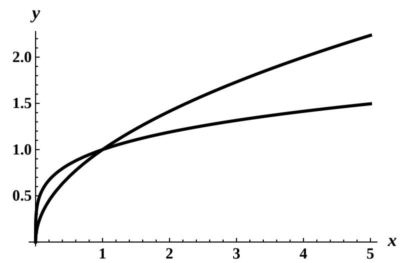
1.2.68



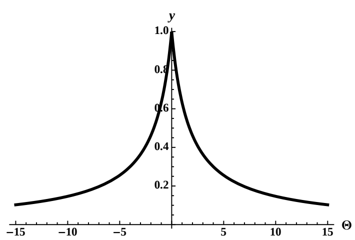
1.2.69



1.2.70



1.2.71

a. 

b. This appears to have a maximum when $\theta = 0$. Our vision is sharpest when we look straight ahead.

c. For $|\theta| \leq .19°$. We have an extremely narrow range where our eyesight is sharp.

1.2.72

a. $f(.75) = \frac{.75^2}{1-2(.75)(.25)} = .9$. There is a 90% chance that the server will win from deuce if they win 75% of their service points.

b. $f(.25) = \frac{.25^2}{1-2(.25)(.75)} = .1$. There is a 10% chance that the server will win from deuce if they win 25% of their service points.

1.2.73

a. Using the points $(1986, 1875)$ and $(2000, 6471)$ we see that the slope is about 328.3. At $t = 0$, the value of p is 1875. Therefore a line which reasonably approximates the data is $p(t) = 328.3t + 1875$.

b. Using this line, we have that $p(9) = 4830$.

1.2.74

a. We know that the points $(32, 0)$ and $(212, 100)$ are on our line. The slope of our line is thus $\frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$. The function $f(F)$ thus has the form $C = (5/9)F + b$, and using the point $(32, 0)$ we see that $0 = (5/9)32 + b$, so $b = -(160/9)$. Thus $C = (5/9)F - (160/9)$

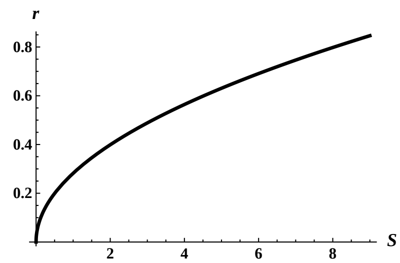
b. Solving the system of equations $C = (5/9)F - (160/9)$ and $C = F$, we have that $F = (5/9)F - (160/9)$, so $(4/9)F = -160/9$, so $F = -40$ when $C = -40$.

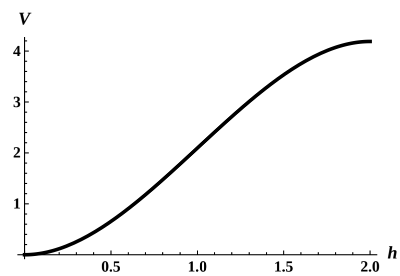
1.2.75

a. Because you are paying \$350 per month, the amount paid after m months is $y = 350m + 1200$.

b. After 4 years (48 months) you have paid $350 \cdot 48 + 1200 = 18000$ dollars. If you then buy the car for \$10,000, you will have paid a total of \$28,000 for the car instead of \$25,000. So you should buy the car instead of leasing it.

1.2.76 Because $S = 4\pi r^2$, we have that $r^2 = \frac{S}{4\pi}$, so $|r| = \frac{\sqrt{S}}{2\sqrt{\pi}}$, but because r is positive, we can write $r = \frac{\sqrt{S}}{2\sqrt{\pi}}$.

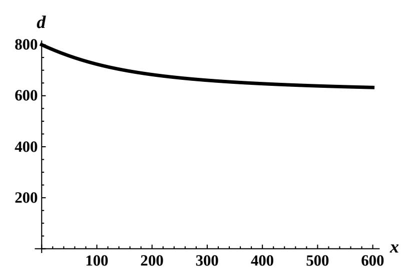


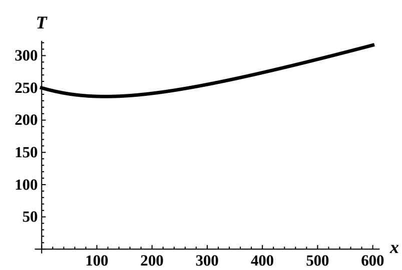


1.2.77 The function makes sense for $0 \leq h \leq 2$.

1.2.78

a. Note that the island, the point P on shore, and the point down shore x units from P form a right triangle. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{40000 + x^2}$. So Kelly must row this distance and then jog $600-x$ meters to get home. So her total distance $d(x) = \sqrt{40000 + x^2} + (600 - x)$.



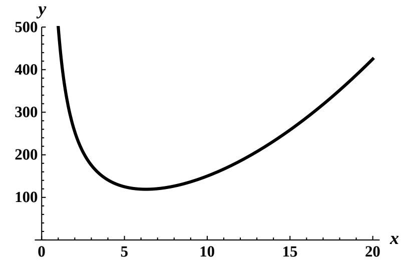


b. Because distance is rate times time, we have that time is distance divided by rate. Thus $T(x) = \frac{\sqrt{40000+x^2}}{2} + \frac{600-x}{4}$.

c. By inspection, it looks as though she should head to a point about 115 meters down shore from P. This would lead to a time of about 236.6 seconds.

1.2.79

a. The volume of the box is x^2h, but because the box has volume 125 cubic feet, we have that $x^2h = 125$, so $h = \frac{125}{x^2}$. The surface area of the box is given by x^2 (the area of the base) plus $4 \cdot hx$, because each side has area hx. Thus $S = x^2 + 4hx = x^2 + \frac{4 \cdot 125 \cdot x}{x^2} = x^2 + \frac{500}{x}$.



b. By inspection, it looks like the value of x which minimizes the surface area is about 6.3.

1.2.80 Let $f(x) = a_n x^n +$ smaller degree terms and let $g(x) = b_m x^m +$ some smaller degree terms.

a. The largest degree term in $f \cdot f$ is $a_n x^n \cdot a_n x^n = a_n^2 x^{n+n}$, so the degree of this polynomial is $n+n = 2n$.

b. The largest degree term in $f \circ f$ involves $a_n \cdot (a_n x^n)^n$, so the degree is n^2.

c. The largest degree term in $f \cdot g$ is $a_n b_m x^{m+n}$, so the degree of the product is $m + n$.

d. The largest degree term in $f \circ g$ involves $a_n \cdot (b_m x^m)^n$, so the degree is mn.

1.2.81 Suppose that the parabola f crosses the x-axis at a and b, with $a < b$. Then a and b are roots of the polynomial, so $(x - a)$ and $(x - b)$ are factors. Thus the polynomial must be $f(x) = c(x - a)(x - b)$ for some non-zero real number c. So $f(x) = cx^2 - c(a + b)x + abc$. Because the vertex always occurs at the x value which is $\frac{-\text{coefficient on x}}{2 \cdot \text{coefficient on } x^2}$ we have that the vertex occurs at $\frac{c(a+b)}{2c} = \frac{a+b}{2}$, which is halfway between a and b.

1.2.82

a. We complete the square to rewrite the function f. Write $f(x) = ax^2 + bx + c$ as $f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a})$. Completing the square yields

$$a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a}\right) + \left(\frac{c}{a} - \frac{b^2}{4a}\right)\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

Thus the graph of f is obtained from the graph of x^2 by shifting $\frac{b}{2a}$ units to the left (and then doing some scaling and vertical shifting) – moving the vertex from 0 to $-\frac{b}{2a}$. The vertex is therefore $\left(\frac{-b}{2a}, c - \frac{b^2}{4}\right)$.

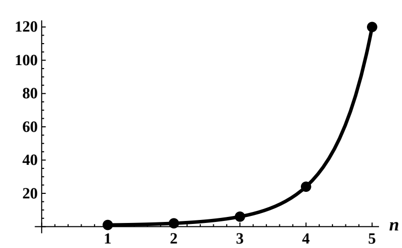
b. We know that the graph of f touches the x-axis twice if the equation $ax^2 + bx + c = 0$ has two real solutions. By the quadratic formula, we know that this occurs exactly when the discriminant $b^2 - 4ac$ is positive. So the condition we seek is for $b^2 - 4ac > 0$, or $b^2 > 4ac$.

1.2.83

a.

n	1	2	3	4	5
$n!$	1	2	6	24	120

b.



c. Using trial and error and a calculator yields that 10! is more than a million, but 9! isn't.

1.2.84

a.

n	1	2	3	4	5	6	7	8	9	10
$S(n)$	1	3	6	10	15	21	28	36	45	55

b. The domain of this function consists of the positive integers. The range is a subset of the set of positive integers.

c. Using trial and error and a calculator yields that $S(n) > 1000$ for the first time for $n = 45$.

1.2.85

a.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	5	14	30	55	91	140	204	285	385

b. The domain of this function consists of the positive integers.

c. Using trial and error and a calculator yields that $T(n) > 1000$ for the first time for $n = 14$.

1.3 Trigonometric Functions

1.3.1 Let O be the length of the side opposite the angle x, let A be length of the side adjacent to the angle x, and let H be the length of the hypotenuse. Then $\sin x = \frac{O}{H}$, $\cos x = \frac{A}{H}$, $\tan x = \frac{O}{A}$, $\csc x = \frac{H}{O}$, $\sec x = \frac{H}{A}$, and $\cot x = \frac{A}{O}$.

1.3.2 We consider the angle formed by the positive x axis and the ray from the origin through the point $P(x, y)$. A positive angle is one for which the rotation from the positive x axis to the other ray is counter-clockwise. We then define the six trigonometric functions as follows: let $r = \sqrt{x^2 + y^2}$. Then $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, and $\cot\theta = \frac{x}{y}$.

1.3.3 The radian measure of an angle θ is the length of the arc s on the unit circle associated with θ.

1.3.4 The period of a function is the smallest positive real number k so that $f(x + k) = f(x)$ for all x in the domain of the function. The sine, cosine, secant, and cosecant function all have period 2π. The tangent and cotangent functions have period π.

1.3.5 $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, and $\tan^2 x + 1 = \sec^2 x$.

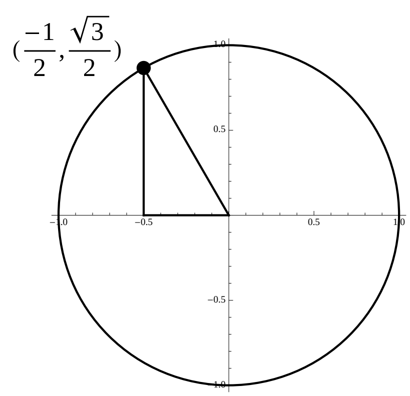
1.3.6 $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{\cos x}{\sin x}$.

1.3.7 The tangent function is undefined where $\cos x = 0$, which is at all real numbers of the form $\frac{\pi}{2} + k\pi, k$ an integer.

1.3.8 $\sec x$ is defined wherever $\cos x \neq 0$, which is $\{x\colon x \neq \frac{\pi}{2} + k\pi, k \text{ an integer}\}$.

1.3.9

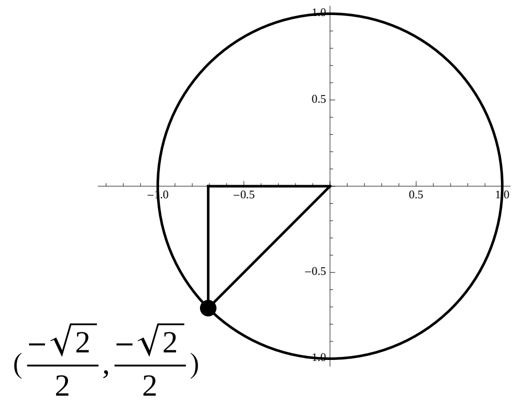
The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\cos(2\pi/3) = -1/2$.



1.3.10 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\sin(2\pi/3) = \sqrt{3}/2$. See the picture from the previous problem.

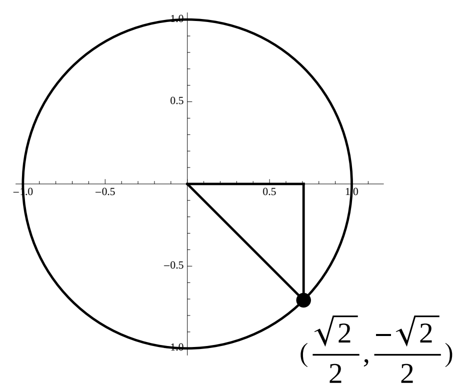
1.3.11

The point on the unit circle associated with $-3\pi/4$ is $(-\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(-3\pi/4) = 1$.



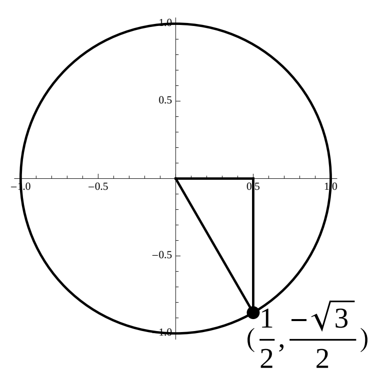
1.3.12

The point on the unit circle associated with $15\pi/4$ is $(\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(15\pi/4) = -1$.



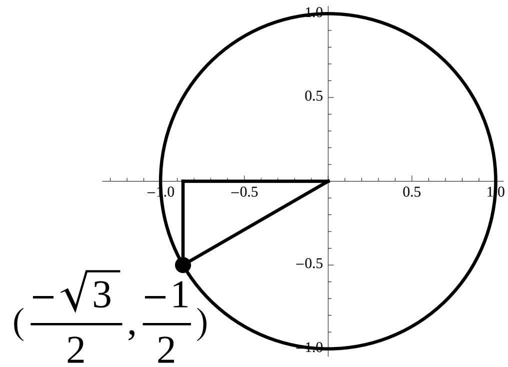
1.3.13

The point on the unit circle associated with $-13\pi/3$ is $(1/2, -\sqrt{3}/2)$, so $\cot(-13\pi/3) = -1/\sqrt{3} = -\sqrt{3}/3$.



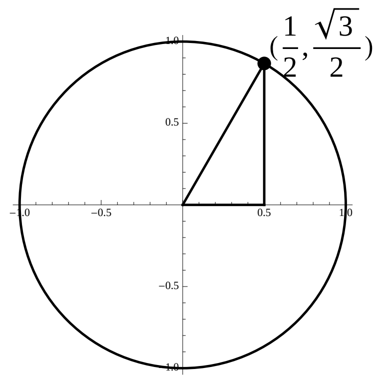
1.3.14

The point on the unit circle associated with $7\pi/6$ is $(-\sqrt{3}/2, -1/2)$, so $\sec(7\pi/6) = -2/\sqrt{3} = -2\sqrt{3}/3$.



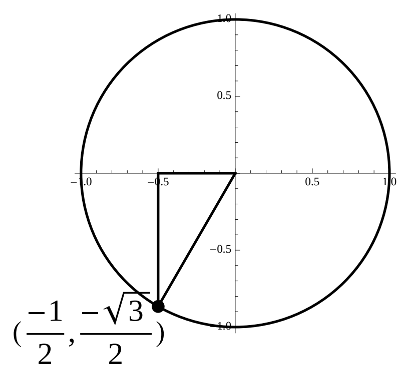
1.3.15

The point on the unit circle associated with $-17\pi/3$ is $(1/2, \sqrt{3}/2)$, so $\cot(-17\pi/3) = 1/\sqrt{3} = \sqrt{3}/3$.



1.3.16

The point on the unit circle associated with $16\pi/3$ is $(-1/2, -\sqrt{3}/2)$, so $\sin(16\pi/3) = -\sqrt{3}/2$.



1.3.17 Because the point on the unit circle associated with $\theta = 0$ is the point $(1, 0)$, we have $\cos 0 = 1$.

1.3.18 Because $-\pi/2$ corresponds to a quarter circle clockwise revolution, the point on the unit circle associated with $-\pi/2$ is the point $(0, -1)$. Thus $\sin(-\pi/2) = -1$.

1.3.19 Because $-\pi$ corresponds to a half circle clockwise revolution, the point on the unit circle associated with $-\pi$ is the point $(-1, 0)$. Thus $\cos(-\pi) = -1$.

1.3.20 Because 3π corresponds to one and a half counterclockwise revolutions, the point on the unit circle associated with 3π is $(-1, 0)$, so $\tan 3\pi = \frac{0}{-1} = 0$.

1.3.21 Because $5\pi/2$ corresponds to one and a quarter counterclockwise revolutions, the point on the unit circle associated with $5\pi/2$ is the same as the point associated with $\pi/2$, which is $(0,1)$. Thus $\sec 5\pi/2$ is undefined.

1.3.22 Because π corresponds to one half circle counterclockwise revolution, the point on the unit circle associated with π is $(-1,0)$. Thus $\cot\pi$ is undefined.

1.3.23 From our definitions of the trigonometric functions via a point $P(x,y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\sec\theta = \frac{r}{x} = \frac{1}{x/r} = \frac{1}{\cos\theta}$.

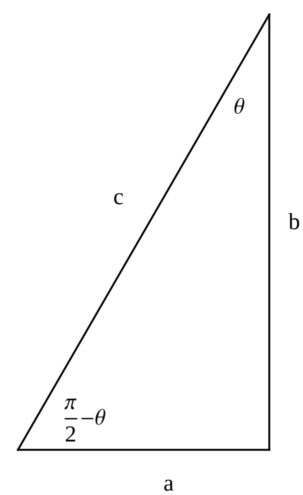
1.3.24 From our definitions of the trigonometric functions via a point $P(x,y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$.

1.3.25 We have already established that $\sin^2\theta + \cos^2\theta = 1$. Dividing both sides by $\cos^2\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$.

1.3.26 We have already established that $\sin^2\theta + \cos^2\theta = 1$. We can write this as $\frac{\sin\theta}{(1/\sin\theta)} + \frac{\cos\theta}{(1/\cos\theta)} = 1$, or $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$.

1.3.27

Using the triangle pictured, we see that $\sec(\pi/2-\theta) = \frac{c}{a} = \csc\theta$.
This also follows from the sum identity $\cos(a+b) = \cos a\cos b - \sin a \sin b$ as follows: $\sec(\pi/2-\theta) = \frac{1}{\cos(\pi/2+(-\theta))} = \frac{1}{\cos(\pi/2)\cos(-\theta)-\sin(\pi/2)\sin(-\theta)} = \frac{1}{0-(-\sin(\theta))} = \csc(\theta)$.



1.3.28 Using the trig identity for the cosine of a sum (mentioned in the previous solution) we have:

$$\sec(x+\pi) = \frac{1}{\cos(x+\pi)} = \frac{1}{\cos(x)\cos(\pi)-\sin(x)\sin(\pi)} = \frac{1}{\cos(x)\cdot(-1)-\sin(x)\cdot 0} = \frac{1}{-\cos(x)} = -\sec x.$$

1.3.29 Using the fact that $\frac{\pi}{12} = \frac{\pi/6}{2}$ and the half-angle identity for cosine:

$$\cos^2(\pi/12) = \frac{1+\cos(\pi/6)}{2} = \frac{1+\sqrt{3}/2}{2} = \frac{2+\sqrt{3}}{4}.$$

Thus, $\cos(\pi/12) = \sqrt{\frac{2+\sqrt{3}}{4}}$.

1.3.30 Using the fact that $\frac{3\pi}{8} = \frac{3\pi/4}{2}$ and the half-angle identities for sine and cosine, we have:

$$\cos^2(3\pi/8) = \frac{1+\cos(3\pi/4)}{2} = \frac{1+(-\sqrt{2}/2)}{2} = \frac{2-\sqrt{2}}{4},$$

and using the fact that $3\pi/8$ is in the first quadrant (and thus has positive value for cosine) we deduce that $\cos(3\pi/8) = \sqrt{2-\sqrt{2}}/2$. A similar calculation using the sine function results in $\sin(3\pi/8) = \sqrt{2+\sqrt{2}}/2$. Thus $\tan(3\pi/8) = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$, which simplifies as

$$\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}} = \sqrt{\frac{(2+\sqrt{2})^2}{2}} = \frac{2+\sqrt{2}}{\sqrt{2}} = 1+\sqrt{2}.$$

1.3.31 First note that $\tan x = 1$ when $\sin x = \cos x$. Using our knowledge of the values of the standard angles between 0 and 2π, we recognize that the sine function and the cosine function are equal at $\pi/4$. Then, because we recall that the period of the tangent function is π, we know that $\tan(\pi/4 + k\pi) = \tan(\pi/4) = 1$ for every integer value of k. Thus the solution set is $\{\pi/4 + k\pi, \text{where } k \text{ is an integer}\}$.

1.3.32 Given that $2\theta\cos(\theta) + \theta = 0$, we have $\theta(2\cos(\theta) + 1) = 0$. Which means that either $\theta = 0$, or $2\cos(\theta) + 1 = 0$. The latter leads to the equation $\cos\theta = -1/2$, which occurs at $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Using the fact that the cosine function has period 2π the entire solution set is thus

$$\{0\} \cup \{2\pi/3 + 2k\pi, \text{where } k \text{ is an integer}\} \cup \{4\pi/3 + 2l\pi, \text{where } l \text{ is an integer}\}.$$

1.3.33 Given that $\sin^2\theta = \frac{1}{4}$, we have $|\sin\theta| = \frac{1}{2}$, so $\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-1}{2}$. It follows that $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$.

1.3.34 Given that $\cos^2\theta = \frac{1}{2}$, we have $|\cos\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Thus $\cos\theta = \frac{\sqrt{2}}{2}$ or $\cos\theta = \frac{-\sqrt{2}}{2}$. We have $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

1.3.35 The equation $\sqrt{2}\sin(x) - 1 = 0$ can be written as $\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Standard solutions to this equation occur at $x = \pi/4$ and $x = 3\pi/4$. Because the sine function has period 2π the set of all solutions can be written as:

$$\{\pi/4 + 2k\pi, \text{where } k \text{ is an integer}\} \cup \{3\pi/4 + 2l\pi, \text{where } l \text{ is an integer}\}.$$

1.3.36 Let $u = 3x$. Note that because $0 \le x < 2\pi$, we have $0 \le u < 6\pi$. Because $\sin u = \sqrt{2}/2$ for $u = \pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, we must have that $\sin 3x = \sqrt{2}/2$ for $3x = \pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, which translates into

$$x = \pi/12, \pi/4, 3\pi/4, 11\pi/12, 17\pi/12, \text{and } 19\pi/12.$$

1.3.37 As in the previous problem, let $u = 3x$. Then we are interested in the solutions to $\cos u = \sin u$, for $0 \le u < 6\pi$.

This would occur for $u = 3x = \pi/4, 5\pi/4, 9\pi/4, 13\pi/4, 17\pi/4, \text{and } 21\pi/4$. Thus there are solutions for the original equation at

$$x = \pi/12, 5\pi/12, 3\pi/4, 13\pi/12, 17\pi/12, \text{and } 7\pi/4.$$

1.3.38 $\sin^2(\theta) - 1 = 0$ wherever $\sin^2(\theta) = 1$, which is wherever $\sin(\theta) = \pm 1$. This occurs for $\theta = \pi/2 + k\pi$, where k is an integer.

1.3.39 If $\sin\theta\cos\theta = 0$, then either $\sin\theta = 0$ or $\cos\theta = 0$. This occurs for $\theta = 0, \pi/2, \pi, 3\pi/2$.

1.3.40 If $\tan^2 2\theta = 1$, then $\sin^2 2\theta = \cos^2 2\theta$, so we have either $\sin 2\theta = \cos 2\theta$ or $\sin 2\theta = -\cos 2\theta$. This occurs for $2\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ for $0 \le 2\theta \le 2\pi$, so the corresponding values for θ are $\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$, $0 \le \theta \le \pi$.

1.3.41

a. False. For example, $\sin(\pi/2 + \pi/2) = \sin(\pi) = 0 \neq \sin(\pi/2) + \sin(\pi/2) = 1 + 1 = 2$.

b. False. That equation has zero solutions, because the range of the cosine function is $[-1, 1]$.

c. False. It has infinitely many solutions of the form $\pi/6 + 2k\pi$, where k is an integer (among others.)

d. False. It has period $\frac{2\pi}{\pi/12} = 24$.

e. True. The others have a range of either $[-1, 1]$ or $(-\infty, -1] \cup [1, \infty)$.

1.3.42 If $\sin\theta = -4/5$, then the Pythagorean identity gives $|\cos\theta| = 3/5$. But if $\pi < \theta < 3\pi/2$, then the cosine of θ is negative, so $\cos\theta = -3/5$. Thus $\tan\theta = 4/3$, $\cot\theta = 3/4$, $\sec\theta = -5/3$, and $\csc\theta = -5/4$.

1.3.43 If $\cos\theta = 5/13$, then the Pythagorean identity gives $|\sin\theta| = 12/13$. But if $0 < \theta < \pi/2$, then the sine of θ is positive, so $\sin\theta = 12/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

1.3.44 If $\sec\theta = 5/3$, then $\cos\theta = 3/5$, and the Pythagorean identity gives $|\sin\theta| = 4/5$. But if $3\pi/2 < \theta < 2\pi$, then the sine of θ is negative, so $\sin\theta = -4/5$. Thus $\tan\theta = -4/3$, $\cot\theta = -3/4$, and $\csc\theta = -5/4$.

1.3.45 If $\csc\theta = 13/12$, then $\sin\theta = 12/13$, and the Pythagorean identity gives $|\cos\theta| = 5/13$. But if $0 < \theta < \pi/2$, then the cosine of θ is positive, so $\cos\theta = 5/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, and $\sec\theta = 13/5$.

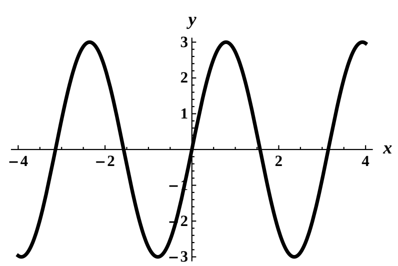
1.3.46 The amplitude is 2, and the period is $\frac{2\pi}{2} = \pi$.

1.3.47 The amplitude is 3, and the period is $\frac{2\pi}{1/3} = 6\pi$.

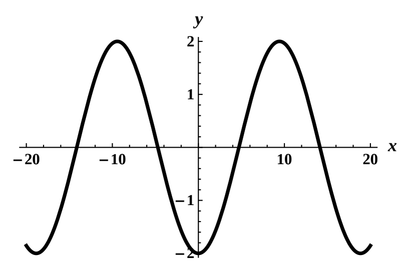
1.3.48 The amplitude is 2.5, and the period is $\frac{2\pi}{1/2} = 4\pi$.

1.3.49 The amplitude is 3.6, and the period is $\frac{2\pi}{\pi/24} = 48$.

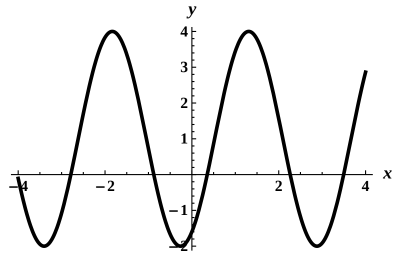
1.3.50



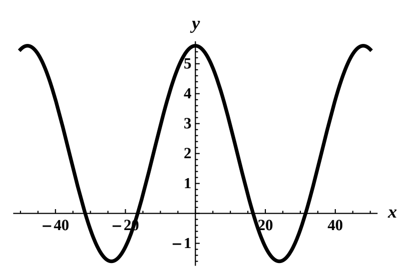
1.3.51



1.3.52



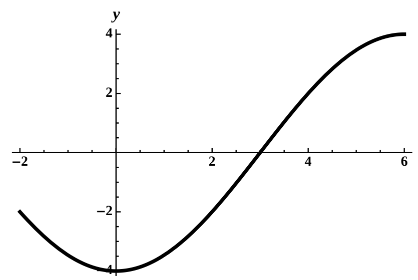
1.3.53



1.3.54

It is helpful to imagine first shifting the function horizontally so that the t intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude. Because the old t-intercept is at $t = 0$ and the new one should be at $t = 3$ (halfway between where the maximum and the minimum occur), we need to shift the function 3 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/6$ so that the new period is $\frac{2\pi}{\pi/6} = 12$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 4. Thus, the desired function is:

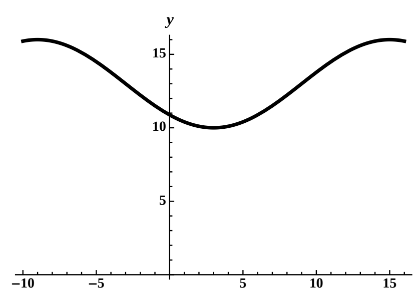
$$f(t) = 4\sin((\pi/6)(t - 3)) = 4\sin((\pi/6)t - \pi/2).$$



1.3.55

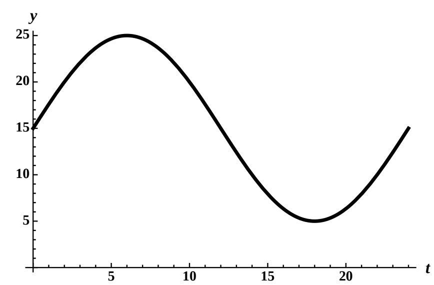
It is helpful to imagine first shifting the function horizontally so that the t intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old t-intercept is at $t = 0$ and the new one should be at $t = 9$ (halfway between where the maximum and the minimum occur), we need to shift the function 9 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\frac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 3, and then shift the whole thing up 13 units. Thus, the desired function is:

$$f(t) = 3\sin((\pi/12)(t-9))+13 = 3\sin((\pi/12)t-3\pi/4)+13.$$



1.3.56

It is helpful to imagine first shifting the function horizontally so that the t intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old t-intercept is at $t = 0$ and the new one should be at $t = 12$ (halfway between where the maximum and the minimum occur), we need to shift the function 12 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\frac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by -10, and then shift the whole thing up 15 units. Thus, the desired function is:



$$f(t) = -10\sin((\pi/12)(t - 12)) + 15.$$

1.3.57 Let C be the circumference of the earth. Then the first rope has radius $r_1 = \frac{C}{2\pi}$. The circle generated by the longer rope has circumference $C + 38$, so its radius is $r_2 = \frac{C+38}{2\pi} = \frac{C}{2\pi} + \frac{38}{2\pi} \approx r_1 + 6$, so the radius of the bigger circle is about 6 feet more than the smaller circle.

1.3.58

a. The period of this function is $\frac{2\pi}{2\pi/365} = 365$.

b. Because the maximum for the regular sine function is 1, and this function is scaled vertically by a factor of 2.8 and shifted 12 units up, the maximum for this function is $(2.8)(1) + 12 = 14.8$. Similarly, the minimum is $(2.8)(-1) + 12 = 9.2$. Because of the horizontal shift, the point at $t = 81$ is the midpoint between where the max and min occur. Thus the max occurs at $81 + (365/4) \approx 172$ and the min occurs approximately $(365/2)$ days later at about $t = 355$.

c. The solstices occur halfway between these points, at 81 and $81 + (365/2) \approx 264$.

1.3.59 We are seeking a function with amplitude 10 and period 1.5, and value 10 at time 0, so it should have the form $10\cos(kt)$, where $\frac{2\pi}{k} = 1.5$. Solving for k yields $k = \frac{4\pi}{3}$, so the desired function is $d(t) = 10\cos(4\pi t/3)$.

1.3.60

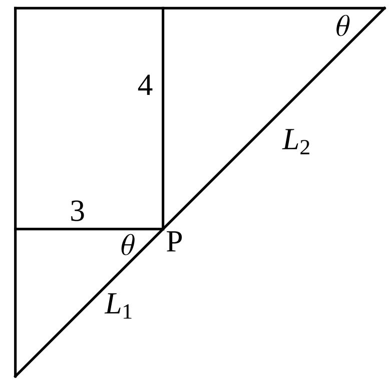
a. Because $\tan\theta = \frac{50}{d}$, we have $d = \frac{50}{\tan\theta}$.

b. Because $\sin\theta = \frac{50}{L}$, we have $L = \frac{50}{\sin\theta}$.

1.3.61 Let L be the line segment connecting the tops of the ladders and let M be the horizontal line segment between the walls h feet above the ground. Now note that the triangle formed by the ladders and L is equilateral, because the angle between the ladders is 60 degrees, and the other two angles must be equal and add to 120, so they are 60 degrees as well. Now we can see that the triangle formed by L, M and the right wall is similar to the triangle formed by the left ladder, the left wall, and the ground, because they are both right triangles with one angle of 75 degrees and one of 15 degrees. Thus $M = h$ is the distance between the walls.

1.3.62

Let the corner point P divide the pole into two pieces, L_1 (which spans the 3-ft hallway) and L_2 (which spans the 4-ft hallway.) Then $L = L_1 + L_2$. Now $L_2 = \frac{4}{\sin\theta}$, and $\frac{3}{L_1} = \cos\theta$ (see diagram.) Thus $L = L_1 + L_2 = \frac{3}{\cos\theta} + \frac{4}{\sin\theta}$. When $L = 10$, $\theta \approx .9273$.



1.3.63

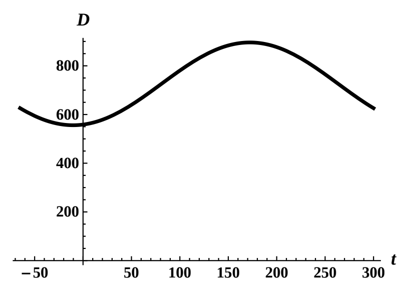
To find $s(t)$ note that we are seeking a periodic function with period 365, and with amplitude 87.5 (which is half of the number of minutes between 7:25 and 4:30). We need to shift the function 4 days plus one fourth of 365, which is about 95 days so that the max and min occur at $t = 4$ days and at half a year later. Also, to get the right value for the maximum and minimum, we need to multiply by negative one and add 117.5 (which represents 30 minutes plus half the amplitude, because $s = 0$ corresponds to 4:00 AM.) Thus we have

$$s(t) = 117.5 - 87.5 \sin\left(\frac{\pi}{182.5}(t - 95)\right).$$

A similar analysis leads to the formula

$$S(t) = 844.5 + 87.5 \sin\left(\frac{\pi}{182.5}(t - 67)\right).$$

The graph pictured shows $D(t) = S(t) - s(t)$, the length of day function, which has its max at the summer solstice which is about the 172nd day of the year, and its min at the winter solstice.



1.3.64 Let θ_1 be the viewing angle to the bottom of the television. Then $\tan\theta_1 = \left(\frac{3}{10}\right)$. Now $\tan(\theta + \theta_1) = \frac{10}{10} = 1$, so $\theta + \theta_1 = \frac{\pi}{4}$, so $\theta = \frac{\pi}{4} - \theta_1 \approx .494$.

1.3.65 The area of the entire circle is πr^2. The ratio $\frac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{r^2\theta}{2}$.

1.3.66 Using the given diagram, drop a perpendicular from the point $(b\cos\theta, b\sin\theta)$ to the x axis, and consider the right triangle thus formed whose hypotenuse has length c. By the Pythagorean theorem, $(b\sin\theta)^2 + (a - b\cos\theta)^2 = c^2$. Expanding the binomial gives $b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta = c^2$. Now because $b^2\sin^2\theta + b^2\cos^2\theta = b^2$, this reduces to $a^2 + b^2 - 2ab\cos\theta = c^2$.

1.3.67 Note that $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$, so $h = c\sin A = a\sin C$. Thus

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Now drop a perpendicular from the vertex A to the line determined by $\overline{BC}$, and let h_2 be the length of this perpendicular. Then $\sin C = \frac{h_2}{b}$ and $\sin B = \frac{h_2}{C}$, so $h_2 = b \sin C = c \sin B$. Thus

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

Putting the two displayed equations together gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

1.4 Chapter One Review

1.4.1

a. True. For example, $f(x) = x^2$ is such a function.

b. False. For example, $\cos(\pi/2 + \pi/2) = \cos(\pi) = -1 \neq \cos(\pi/2) + \cos(\pi/2) = 0 + 0 = 0$.

c. False. Consider $f(1+1) = f(2) = 2m + b \neq f(1) + f(1) = (m+b) + (m+b) = 2m + 2b$. (At least these aren't equal when $b \neq 0$.)

d. True. $f(f(x)) = f(1-x) = 1 - (1-x) = x$.

e. False. This set is the union of the disjoint intervals $(-\infty, -7)$ and $(1, \infty)$.

1.4.2

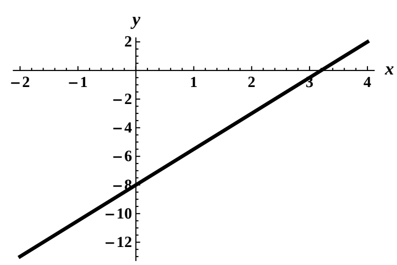
a. Because the quantity under the radical must be non-zero, the domain of f is $[0, \infty)$. The range is also $[0, \infty)$.

b. The domain is $(-\infty, 2) \cup (2, \infty)$. The range is $(-\infty, 0) \cup (0, \infty)$. (Note that if 0 were in the range then $\frac{1}{y-2} = 0$ for some value of y, but this expression has no real solutions.)

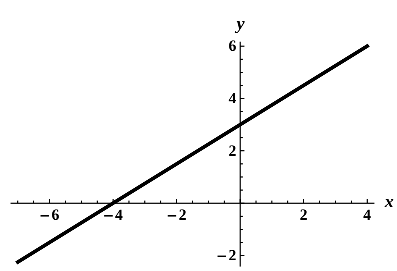
c. Because h can be written $h(z) = \sqrt{(z-3)(z+1)}$, we see that the domain is $(-\infty, -1] \cup [3, \infty)$. The range is $[0, \infty)$. (Note that as z gets large, $h(z)$ gets large as well.)

1.4.3

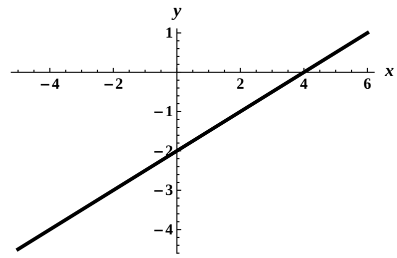
a. This line has slope $\frac{2-(-3)}{4-2} = 5/2$. Therefore the equation of the line is $y - 2 = \frac{5}{2}(x-4)$, so $y = \frac{5}{2}x - 8$.



b. This line has the form $y = (3/4)x + b$, and because $(-4, 0)$ is on the line, $0 = (3/4)(-4) + b$, so $b = 3$. Thus the equation of the line is given by $y = (3/4)x + 3$.

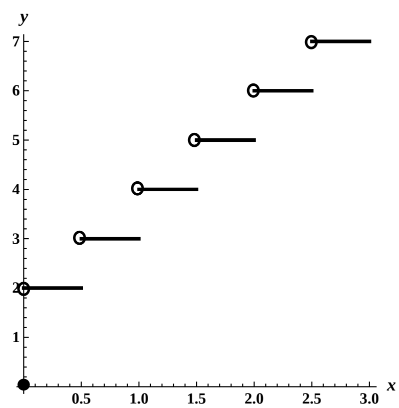


c. This line has slope $\frac{0-(-2)}{4-0} = \frac{1}{2}$, and the y-intercept is given to be -2, so the equation of this line is $y = (1/2)x - 2$.



1.4.4

The function is a piecewise step function which jumps up by one every half-hour step.

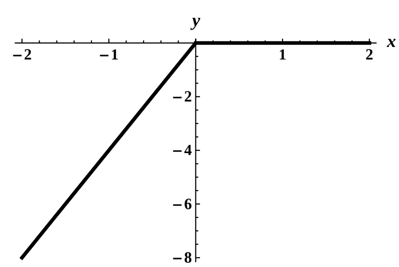


1.4.5

Because $|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x \geq 0, \end{cases}$

we have

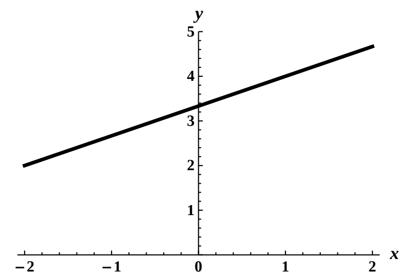
$2(x - |x|) = \begin{cases} 2(x - (-x)) = 4x & \text{if } x < 0; \\ 2(x - x) = 0 & \text{if } x \geq 0. \end{cases}$



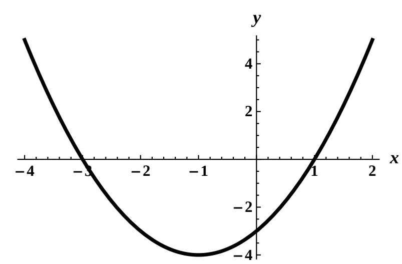
1.4.6 Because the trip is 500 miles in a car that gets 35 miles per gallon, $\frac{500}{35} = \frac{100}{7}$ represents the number of gallons required for the trip. If we multiply this times the number of dollars per gallon we will get the cost. Thus $C = f(p) = \frac{100}{7}p$ dollars.

1.4.7

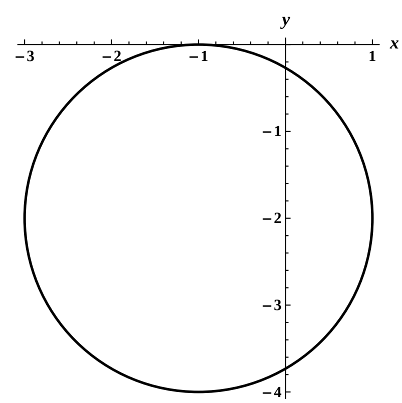
a. This is a straight line with slope $2/3$ and y-intercept $10/3$.



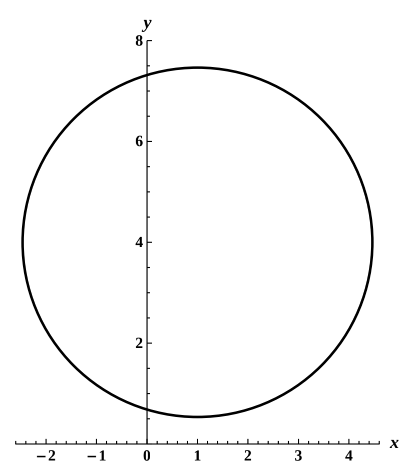
b. Completing the square gives $y = (x^2+2x+1)-4$, or $y = (x+1)^2-4$, so this is the standard parabola shifted one unit to the left and down 4 units.



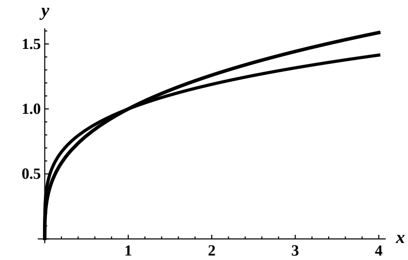
c. Completing the square, we have $x^2+2x+1+y^2+4y+4 = -1+1+1+4$, so we have $(x+1)^2+(y+2)^2 = 4$, a circle of radius 2 centered at $(-1,-2)$.



d. Completing the square, we have $x^2-2x+1+y^2-8y+16 = -5+1+16$, or $(x-1)^2+(y-4)^2 = 12$, which is a circle of radius $\sqrt{12}$ centered at $(1,4)$.



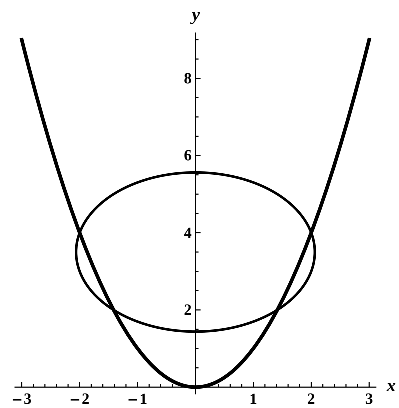
1.4.8 To solve $x^{1/3} = x^{1/4}$ we raise each side to the 12th power, yielding $x^4 = x^3$. This gives $x^4 - x^3 = 0$, or $x^3(x-1) = 0$, so the only solutions are $x = 0$ and $x = 1$ (which can be easily verified as solutions.) Between 0 and 1, $x^{1/4} > x^{1/3}$, but for $x > 1$, $x^{1/3} > x^{1/4}$.



1.4.9 The domain of $x^{1/7}$ is the set of all real numbers, as is its range. The domain of $x^{1/4}$ is the set of non-negative real numbers, as is its range.

1.4.10

Completing the square in the second equation, we have $x^2 + y^2 - 7y + \frac{49}{4} = -8 + \frac{49}{4}$, which can be written as $x^2 + (y - (7/2))^2 = \frac{17}{4}$. Thus we have a circle of radius $\sqrt{17}/2$ centered at $(0, 7/2)$, along with the standard parabola. These intersect when $y = 7y - y^2 - 8$, which occurs for $y^2 - 6y + 8 = 0$, so for $y = 2$ and $y = 4$, with corresponding x values of ± 2 and $\pm\sqrt{2}$.



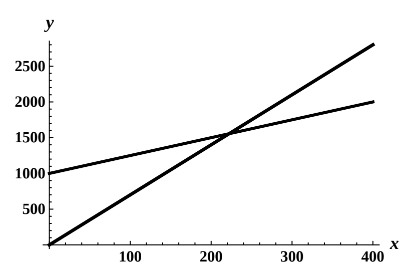
1.4.11 We are looking for the line between the points $(0, 212)$ and $(6000, 200)$. The slope is $\frac{212-200}{0-6000} = -12/6000 = -1/500$. Because the intercept is given, we deduce that the line is $B = f(a) = (-1/500)(a) + 212$.

1.4.12

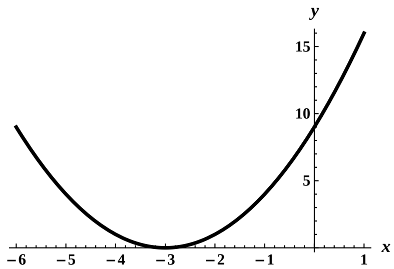
a. The cost of producing x books is $C(x) = 1000 + 2.5x$.

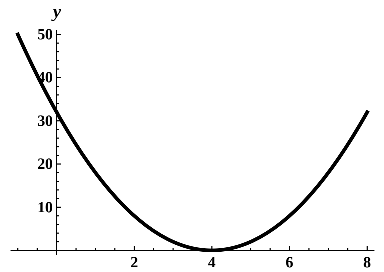
b. The revenue generated by selling x books is $R(x) = 7x$.

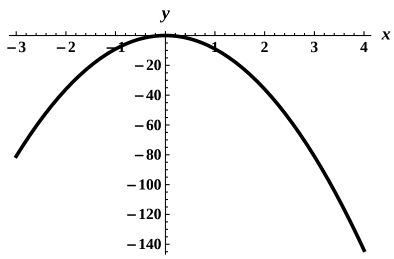
c. The break-even point is where $R(x) = C(x)$. This is where $7x = 1000 + 2.5x$, or $4.5x = 1000$. So $x = \frac{1000}{4.5} \approx 222$.

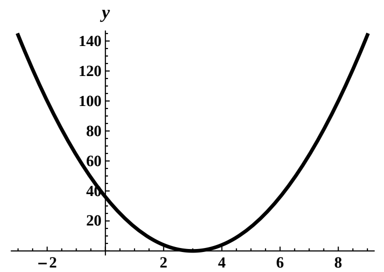


1.4.13

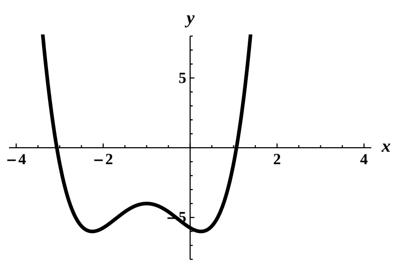
a. 

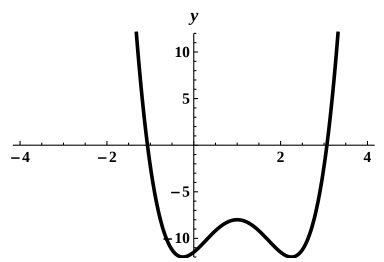
b. 

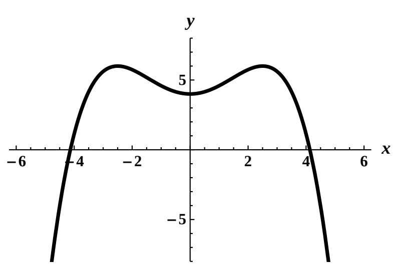
c. 

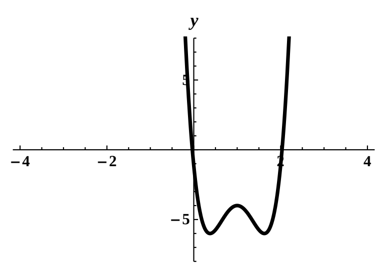
d. 

1.4.14

a. 

b. 

c. 

d. 

1.4.15

a. $h(g(\pi/2)) = h(1) = 1$

b. $h(f(x)) = h(x^3) = x^{3/2}$.

c. $f(g(h(x))) = f(g(\sqrt{x})) = f(\sin(\sqrt{x})) = (\sin(\sqrt{x}))^3$.

d The domain of $g(f(x))$ is $\mathbb{R}$, because the domain of both functions is the set of all real numbers.

e. The range of $f(g(x))$ is $[-1, 1]$. This is because the range of g is $[-1, 1]$, and on the restricted domain $[-1, 1]$, the range of f is also $[-1, 1]$.

1.4.16

a. If $g(x) = x^2 + 1$ and $f(x) = \sin x$, then $f(g(x)) = f(x^2+1) = \sin(x^2+1)$.

b. If $g(x) = x^2 - 4$ and $f(x) = x^{-3}$ then $f(g(x)) = f(x^2-4) = (x^2-4)^{-3}$.

1.4.17 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-2(x+h)-(x^2-2x)}{h} = \frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h} = \frac{2hx+h^2-2h}{h} = 2x+h-2.$

$\frac{f(x)-f(a)}{x-a} = \frac{x^2-2x-(a^2-2a)}{x-a} = \frac{(x^2-a^2)-2(x-a)}{x-a} = \frac{(x-a)(x+a)-2(x-a)}{x-a} = x+a-2.$

1.4.18 $\frac{f(x+h)-f(x)}{h} = \frac{4-5(x+h)-(4-5x)}{h} = \frac{4-5x-5h-4+5x}{h} = \frac{-5h}{h} = -5.$

$\frac{f(x)-f(a)}{x-a} = \frac{4-5x-(4-5a)}{x-a} = \frac{-5(x-a)}{x-a} = -5.$

1.4.19 $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+2-(x^3+2)}{h} = \frac{x^2+3x^2h+3xh^2+h^3+2-x^3-2}{h} = \frac{h(3x^2+3xh+h^2)}{h} = 3x^2+3xh+h^2.$

$\frac{f(x)-f(a)}{x-a} = \frac{x^3+2-(a^3+2)}{x-a} = \frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2+ax+a^2.$

1.4.20 $\frac{f(x+h)-f(x)}{h} = \frac{\frac{7}{x+h+3}-\frac{7}{x+3}}{h} = \frac{\frac{7x+21-(7x+7h+21)}{(x+3)(x+h+3)}}{h} = \frac{-7h}{(h)(x+3)(x+h+3)} = \frac{-7}{(x+3)(x+h+3)}.$

$\frac{f(x)-f(a)}{x-a} = \frac{\frac{7}{x+3}-\frac{7}{a+3}}{x-a} = \frac{\frac{7a+21-(7x+21)}{(x+3)(a+3)}}{x-a} = \frac{-7(x-a)}{(x-a)(x+3)(a+3)} = \frac{-7}{(x+3)(a+3)}.$

1.4.21

a. Because $f(-x) = \cos(-3x) = \cos 3x = f(x)$, this is an even function, and is symmetric about the y-axis.

b. Because $f(-x) = 3(-x)^4 - 3(-x)^2 + 1 = 3x^4 - 3x^2 + 1 = f(x)$, this is an even function, and is symmetric about the y-axis.

c. Because replacing x by $-x$ and/or replacing y by $-y$ gives the same equation, this represents a curve which is symmetric about the y-axis and about the origin and about the x-axis.

1.4.22

a. $\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{(\sin\theta)(1-\cos\theta)} = \frac{\sin^2\theta}{(\sin\theta)(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}.$

b. $\frac{\sec\theta-1}{\tan\theta} = \frac{\sec\theta-1}{\tan\theta}\cdot\frac{\sec\theta+1}{\sec\theta+1} = \frac{\sec^2\theta-1}{(\tan\theta)(\sec^2\theta+1)} = \frac{\tan^2\theta}{(\tan\theta)(\sec^2\theta+1)} = \frac{\tan\theta}{\sec^2\theta+1}.$

1.4.23

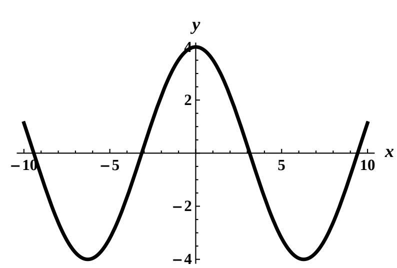
a. A 135 degree angle measures $135\cdot(\pi/180)$ radians, which is $3\pi/4$ radians.

b. A $4\pi/5$ radian angle measues $4\pi/5\cdot(180/\pi)$ degrees, which is 144 degrees.

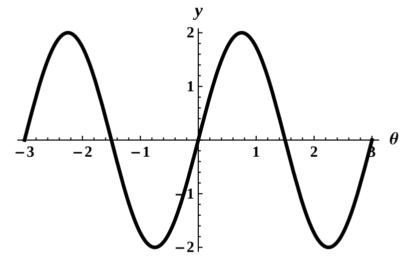
c. Because the length of the arc is the measure of the subtended angle (in radians) times the radius, this arc would be $4\pi/3\cdot 10 = \frac{40\pi}{3}$ units long.

1.4.24

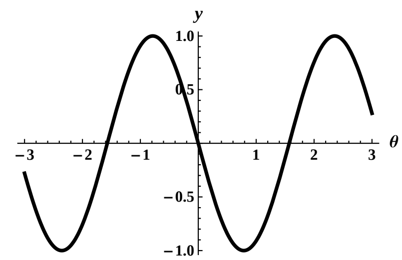
a. This function has period $\frac{2\pi}{1/2} = 4\pi$ and amplitude 4.



b. This function has period $\frac{2\pi}{2\pi/3} = 3$ and amplitude 2.



c. This function has period $\frac{2\pi}{2} = \pi$ and amplitude 1. Compared to the ordinary cosine function it is compressed horizontally, flipped about the x-axis, and shifted $\pi/4$ units to the right.



1.4.25

a. We need to scale the ordinary cosine function so that its period is 6, and then shift it 3 units to the right, and multiply it by 2. So the function we seek is $y = 2\cos((\pi/3)(t-3))$.

b. We need to scale the ordinary cosine function so that its period is 24, and then shift it to the right 6 units. We then need to change the amplitude to be half the difference between the maximum and minimum, which would be 5. Then finally we need to shift the whole thing up by 15 units. The function we seek is thus $y = 15 + 5\cos((\pi/12)(t-6))$. This can also be written $15 + 5\sin(\pi t/12)$.

1.4.26 The pictured function has a period of π, an amplitude of 2, and a maximum of 3 and a minimum of -1. It can be described by $y = 1 + 2\cos(2(x - \pi/2))$.

1.4.27

a. $-\sin x$ is pictured in F.

b. $\cos 2x$ is pictured in E.

c. $\tan(x/2)$ is pictured in D.

d. $-\sec x$ is pictured in B.

e. $\cot 2x$ is pictured in C.

f. $\sin^2 x$ is pictured in A.

1.4.28 If $\sec x = 2$, then $\cos x = \frac{1}{2}$. This occurs for $x = -\pi/3$ and $x = \pi/3$, so the intersection points are $(-\pi/3, 2)$ and $(\pi/3, 2)$.

1.4.29 $\sin x = \frac{-1}{2}$ for $x = 7\pi/6$ and for $x = 11\pi/6$, so the intersection points are $(7\pi/6, -1/2)$ and $(11\pi/6, -1/2)$.

1.4.30 Let N be the north pole, and C the center of the given circle, and consider the angle CNP. This angle measures $\frac{\pi-\varphi}{2}$. (Note that the triangle CNP is isosceles.) Now consider the triangle NOX where O is the origin and X is the point $(x, 0)$. Using triangle NOX, we have $\tan\left(\frac{\pi-\varphi}{2}\right) = \frac{x}{2R}$, so $x = 2R\tan\left(\frac{\pi-\varphi}{2}\right)$.

Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time $t = a$ and $t = b$ is the change in position divided by the elapsed time: $v_{\text{av}} = \frac{s(b)-s(a)}{b-a}$.

2.1.2 In order to compute the instantaneous velocity of the object at time $t = a$, we compute the average velocity over smaller and smaller time intervals of the form $[a, t]$, using the formula: $v_{\text{av}} = \frac{s(t)-s(a)}{t-a}$. We let t approach a. If the quantity $\frac{s(t)-s(a)}{t-a}$ approaches a limit as $t \to a$, then that limit is called the instantaneous velocity of the object at time $t = a$.

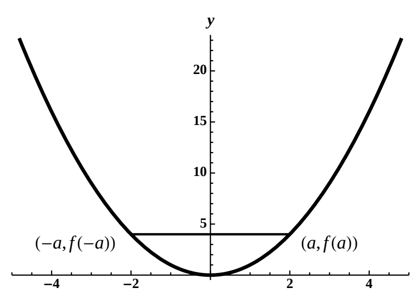
2.1.3 The slope of the secant line between points $(a, f(a))$ and $(b, f(b))$ is the ratio of the differences $f(b) - f(a)$ and $b - a$. Thus $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a}$.

2.1.4 In order to compute the slope of the tangent line to the graph of $y = f(t$ at $(a, f(a))$, we compute the slope of the secant line over smaller and smaller time intervals of the form $[a, t]$. Thus we consider $\frac{f(t)-f(a)}{t-a}$ and let $t \to a$. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve $y = f(t)$ at $t = a$.

2.1.5 Both problems involve the same mathematics, namely finding the limit as $t \to a$ of a quotient of differences of the form $\frac{g(t)-g(a)}{t-a}$ for some function g.

2.1.6

Because $f(x) = x^2$ is an even function, $f(-a) = f(a)$ for all a. Thus the slope of the secant line between the points $(a, f(a))$ and $(-a, f(-a))$ is $m_{\text{sec}} = \frac{f(-a)-f(a)}{-a-a} = \frac{0}{-2a} = 0$. The slope of the tangent line at $x = 0$ is also zero.



2.1.7 The average velocity is $\frac{s(3)-s(2)}{3-2} = 156 - 136 = 20$.

2.1.8 The average velocity is $\frac{s(4)-s(1)}{4-1} = \frac{144-84}{3} = \frac{60}{3} = 20$.

2.1.9

a. Over $[1,4]$, we have $v_{\text{av}} = \frac{s(4)-s(1)}{4-1} = \frac{256-112}{3} = 48$.

b. Over $[1,3]$, we have $v_{\text{av}} = \frac{s(3)-s(1)}{3-1} = \frac{240-112}{2} = 64$.

c. Over $[1,2]$, we have $v_{\text{av}} = \frac{s(2)-s(1)}{2-1} = \frac{192-112}{1} = 80$.

d. Over $[1,1+h]$, we have $v_{\text{av}} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{-16(1+h)^2+128(1+h)-(112)}{h} = \frac{-16h^2-32h+128h}{h} = \frac{h(-16h+96)}{h} = 96-16h = 16(6-h)$.

2.1.10

a. Over $[0,3]$, we have $v_{\text{av}} = \frac{s(3)-s(0)}{3-0} = \frac{65.9-20}{3} = 15.3$.

b. Over $[0,2]$, we have $v_{\text{av}} = \frac{s(2)-s(0)}{2-0} = \frac{60.4-20}{2} = 20.2$.

c. Over $[0,1]$, we have $v_{\text{av}} = \frac{s(1)-s(0)}{1-0} = \frac{45.1-20}{1} = 25.1$.

d. Over $[0,h]$, we have $v_{\text{av}} = \frac{s(h)-s(0)}{h-0} = \frac{-4.9h^2+30h+20-20}{h} = \frac{(h)(-4.9h+30)}{h} = -4.9h+30$.

2.1.11

a. $\frac{s(2)-s(0)}{2-0} = \frac{72-0}{2} = 36$.

b. $\frac{s(1.5)-s(0)}{1.5-0} = \frac{66-0}{1.5} = 44$.

c. $\frac{s(1)-s(0)}{1-0} = \frac{52-0}{1} = 52$.

d. $\frac{s(.5)-s(0)}{.5-0} = \frac{30-0}{.5} = 60$.

2.1.12

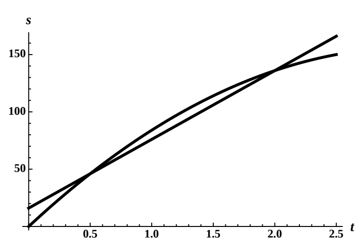
a. $\frac{s(2.5)-s(.5)}{2.5-.5} = \frac{150-46}{2} = 52$.

b. $\frac{s(2)-s(.5)}{2-.5} = \frac{136-46}{1.5} = 60$.

c. $\frac{s(1.5)-s(.5)}{1.5-.5} = \frac{114-46}{1} = 68$.

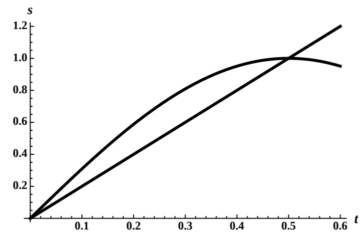
d. $\frac{s(1)-s(.5)}{1-.5} = \frac{84-46}{.5} = 76$.

2.1.13



The slope of the secant line is given by $\frac{s(2)-s(.5)}{2-.5} = \frac{136-46}{1.5} = 60$. This represents the average velocity of the object over the time interval $[.5,2]$.

2.1.14



The slope of the secant line is given by $\frac{s(.5)-s(0)}{.5-0} = \frac{1}{.5} = 2$. This represents the average velocity of the object over the time interval $[0, .5]$.

2.1.15

Time Interval	$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.16

Time Interval	$[2, 3]$	$[2, 2.25]$	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.17 $\frac{s(1.01)-s(1)}{.01} = 47.84$, while $\frac{s(1.001)-s(1)}{.001} = 47.984$ and $\frac{s(1.0001)-s(1)}{.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.18 $\frac{s(2.01)-s(2)}{.01} = -4.16$, while $\frac{s(2.001)-s(2)}{.001} = -4.016$ and $\frac{s(2.0001)-s(2)}{.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.19

Time Interval	$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[2.99999, 3]$
Average Velocity	20	5.6	4.16	4.016	4.0016	4.00016

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	$[\pi/2, \pi]$	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-1.90986	$-.149875$	$-.0149999$	$-.0015$	$-.00015$

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
Average Velocity	-17.6	-16.16	-16.016	-16.002

The instantaneous velocity appears to be -16 ft/s.

2.1.22

Time Interval	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-19.9667	-19.9997	-20.0000	-20.0000

The instantaneous velocity appears to be -20 ft/s.

2.1.23

Time Interval	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Average Velocity	79.4677	79.9947	79.9999	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Average Velocity	-10	-18.1818	-19.802	-19.98

The instantaneous velocity appears to be -20 ft/s.

2.1.25

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x **Interval**	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Slope of Secant Line	-2.995	-2.99995	-3.0000	-3.0000

The slope of the tangent line appears to be -3.

2.1.27

x **Interval**	$[-1, -.9]$	$[-1, -.99]$	$[-1, -.999]$	$[-1, -.9999]$
Slope of the Secant Line	.524862	.5025	.50025	.500025

The slope of the tangent line appears to be .5.

2.1.28

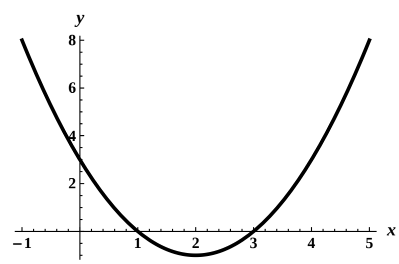
x **Interval**	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex $(2, -1)$.

b. At $(2, -1)$ the function has tangent line with slope 0.



c.

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of the Secant Line	.1	.01	.001	.0001

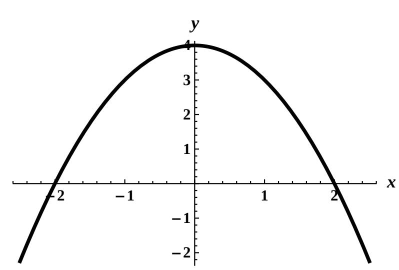
The slope of the tangent line at $(2, -1)$ appears to be 0.

2.1.30

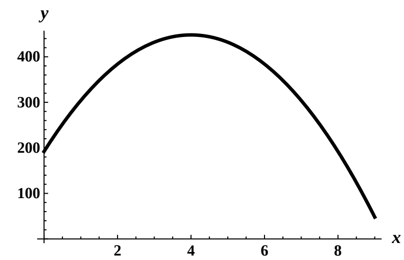
a. Note that the graph is a parabola with vertex $(0, 4)$.

b. At $(0, 4)$ the function has a tangent line with slope 0.

c. This is true for this function – because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus $f(0 + h) = 4 - (0 + h)^2 = 4 - (-h)^2 = f(0 - h)$. So the slope of any such secant line is $\frac{4-h^2-(4-h^2)}{h-(-h)} = \frac{0}{2h} = 0$.



2.1.31



a. Note that the graph is a parabola with vertex $(4, 448)$.

b. At $(4, 448)$ the function has tangent line with slope 0, so $a = 4$.

c.

x **Interval**	$[4, 4.1]$	$[4, 4.01]$	$[4, 4.001]$	$[4, 4.0001]$
Slope of the Secant Line	-1.6	$-.16$	$-.016$	$-.0016$

The slopes of the secant lines appear to be approaching zero.

d. On the interval $[0, 4)$ the instantaneous velocity of the projectile is positive.

e. On the interval $(4, 9]$ the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when $s(t) = 96$. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

b.

t **Interval**	$[\sqrt{6} - .1, \sqrt{6}]$	$[\sqrt{6} - .01, \sqrt{6}]$	$[\sqrt{6} - .001, \sqrt{6}]$	$[\sqrt{6} - .0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx .63662.$$

For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + .5) - f(\pi/2)}{(\pi/2 + .5) - (\pi/2)} = \frac{-\cos(\pi/2 + .5)}{.5} \approx .958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + .05) - f(\pi/2)}{(\pi/2 + .05) - (\pi/2)} = \frac{-\cos(\pi/2 + .05)}{.05} \approx .999583.$$

Computing one more slope of a secant line:

$$m_{\text{sec}} = \frac{f(\pi/2 + .01) - f(\pi/2)}{(\pi/2 + .01) - (\pi/2)} = \frac{-\cos(\pi/2 + .01)}{.01} \approx .999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definitions of Limits

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3 Suppose the function f is defined for all x near a but greater than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then $\lim_{x\to a^+} f(x) = L$.

2.2.4 Suppose the function f is defined for all x near a but less than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then $\lim_{x\to a^-} f(x) = L$.

2.2.5 It must be true that $L = M$.

2.2.6 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

2.2.7

a. $h(2) = 5$.

b. $\lim_{x\to 2} h(x) = 3$.

c. $h(4)$ does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.8

a. $g(0) = 0$.

b. $\lim_{x\to 0} g(x) = 1$.

c. $g(1) = 2$.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.9

a. $f(1) = -1$.

b. $\lim_{x\to 1} f(x) = 1$.

c. $f(0) = 2$.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.10

a. $f(2) = 2$.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.11

a.

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x) = \frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999	undefined	4.0001	4.001	4.01	4.1

b. $\lim_{x\to 2} f(x) = 4$.

2.2.12

a.

x	.9	.99	.999	.9999	1	1.0001	1.001	1.01	1.1
$f(x) = \frac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	undefined	3.0003	3.003	3.0301	3.31

b. $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = 3$